COMPLETE SYMMETRIC VARIETIES

by

C. De Concini and C. Procesi

Università di Roma II — Università di Roma

Nur der Philister schwärmt für absolute Symmetrie

H. Seidel, ges. w. 1,70 *

INTRODUCTION

In the study of enumerative problems on plane conics the following variety has been extensively studied ([6],[7],[15],[17],[18],[19],[20],[23],[25]).

We consider pairs (C,C') where C is a non degenerate conic and C' its dual and call X the closure of this correspondence in the variety of pairs of conics in $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$.

On this variety acts naturally the projective group of the plane and one can see that X decomposes into 4 orbits: X_o open in X; X_1, X_2 of codimension 1 and $X_3 = \bar{X}_1 \cap \bar{X}_2$ of codimension 2. All orbit closures are smooth and the intersection of $\bar{X}_1$ with $\bar{X}_2$ is transversal. This theory has been extended to higher dimensional quadrics ([1],[15],[17],[21]) and also carried out in the similar example of collineations ([16]).

The renewed interest in enumerative geometry (see e.g. [11]) has brought back some interest in this class of varieties ([22],[5] , cf.§6).

In this paper we will study closely a general class of varieties, including the previous examples, which have a significance for enumerative problems.

Let $\bar{G}$ be a semisimple adjoint group, $\sigma: \bar{G} \to \bar{G}$ an automorphism of order 2 and $\bar{H} = \bar{G}^{\sigma}$. We construct a canonical variety X with an action of $\bar{G}$ such that

1) X has an open orbit isomorphic to $\bar{G}/\bar{H}$
2) X is smooth with finitely many $\bar{G}$ orbits
3) The orbit closures are all smooth
4) There is a 1-1 correspondence between the set of orbit closures and the family of subsets of a set I_ℓ with ℓ elements. If $J \subseteq I_\ell$ we denote by S_J the corresponding orbit closure
5) We have $S_I \cap S_J = S_{I \cup J}$ and codim S_I = card I

* We thank the "Lessico intellettuale europeo" for supplying the quotation.

6) Each S_I is the transversal complete intersection of the $S_{(u)}$, $u \in I$

7) For each S_I we have a $\bar{G}$ equivariant fibration $\pi_I : S_I \to G/P_I$ with P_I a parabolic subgroup with semisimple Levi factor L, σ stable, and the fiber of π_I is the canonical projective variety associated to L and $\sigma|L$

Using results of Bialynicki Birula [2] we give a paving of X by affine spaces and compute its Picard group. We describe the positive line bundles on X and their cohomology in a fashion similar to that of "Flag varieties".

Next we give a precise algorithm which allows to compute the so called characteristic numbers of basic conditions (in the classical terminology) in all cases. The computation can be carried out mechanically although it is very lengthy.

As an example we give the classical application due to H.Schubert [14] for space quadrics and compute the number of quadrics tangent to nine quadrics in general position.

We should now make three final remarks. First of all our method has been strongly influenced by the work of Semple [15], we have in fact interpeted his construction in the language of algebraic groups. The second point will be taken in a continuation of this work.Briefly we should say that a general theory of group embeddings due to Luna and Vust [13] has been used by Vust to classify all projective equivariant embeddings of a symmetric variety of adjoint type and in particular the ones which have the property that each orbit closure is smooth. We call such embeddings wonderful. It has been shown by Vust that such embeddings are all obtained in most cases from our variety X by successive blow ups, followed by a suitable contraction.

This is the reason why we sometimes refer to X as the minimal compactification, in fact it is minimal only among this special class.

The study of the limit provariety obtained in this way is the clue for a general understanding of enumerative questions on symmetric varieties as we plan to show elsewhere.

Finally we have restricted our analysis to characteristic 0 for simplicity. Many of our results are valid in all characteristics (with the possible exception of 2) and some should have a suitable characteristic free analogue. Hopefully an analysis of this theory may have same applications to representation theory also in positive characteristic.

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1. PRELIMINARIES

In this section we collect a few more or less well known facts.

1.1. Let G be a semisimple simply connected algebraic group over the complex numbers. Let $\sigma: G \to G$ be an automorphism of order 2 and $H = G^\sigma$ the subgroup of G of the elements fixed under σ. The homogeneous space G/H is by definition a symmetric variety and more generally, if G' is a quotient of G by a (finite) σ stable subgroup of the center of G, the corresponding G'/H' will again be a symmetric variety.

Let $\underline{g}$, $\underline{h}$ denote the Lie algebras of G, H respectively. σ induces an automorphism of order 2 in $\underline{g}$ which will again be denoted by σ and $\underline{h}$ is exactly the +1 eigenspace of σ.

We recall a well known fact:

PROPOSITION. Every σ-stable torus in G is contained in a maximal torus of G which is σ stable.

If T is a σ stable torus and $\underline{t}$ its Lie algebra, we can decompose $\underline{t}$ as $\underline{t} = \underline{t}_o \oplus \underline{t}_1$ according to the eigenvalues +1, -1 of σ. $\underline{t}_o$ is the Lie algebra of the torus $T_o = T^\sigma$ while $\underline{t}_1$ is the Lie algebra of the torus $T_1 = \{t \in T \mid t^\sigma = t^{-1}\}$ such a torus is called anisotropic. The natural mapping $T_o \times T_1 \to T$ is an isogeny, it is not necessarily an isomorphism since the character group of T need not decompose under σ into the sum of the subgroups relative to the eigenvalues ±1. We indicate still by σ the induced mapping on $\underline{t}^*$ and can easily verify in case T is a maximal torus and $\Phi \subseteq \underline{t}^*$ the root system:

i) If $\underline{t} \oplus \sum_{\alpha\in\Phi} \underline{g}_\alpha$ is the root space decomposition of $\underline{g}$ then $\sigma(\underline{g}_\alpha) = \underline{g}_{\alpha\sigma}$, hence $\sigma(\Phi) = \Phi$.

(ii) σ preserves the Killing form.

We want now to choose among all possible σ stable tori one for which dim T_1 is maximal and call this dimension the rank of G/H, indicated by ℓ.

1.2. Having fixed T and so the root system Φ we proceed now to fix the positive roots in a compatible way.

LEMMA. One can choose the set Φ^+ of positive roots in such a way that: If $\alpha \in \Phi^+$ and $\alpha \not\equiv 0$ on $\underline{t}_1$ then $\alpha^\sigma \in \Phi^-$.

PROOF. Decompose $\underline{t}^* = \underline{t}_o^* \oplus \underline{t}_1^*$; every root α is then written $\alpha = \alpha_o + \alpha_1$ and $\alpha^\sigma = \alpha_o - \alpha_1$. Choose two R-linear forms ϕ_o and ϕ_1 on $\underline{t}_o^*$ and $\underline{t}_1^*$ such that ϕ_o and ϕ_1 are non zero on the non zero components of the roots. We can replace ϕ_1 by a multiple if necessary so that, if $\alpha = \alpha_o + \alpha_1$ and $\alpha_1 \neq 0$ we have $|\phi_1(\alpha_1)| > |\phi_o(\alpha_o)|$. Consider now the R-linear form $\phi = \phi_o \oplus \phi_1$, we have that $\phi(\alpha) \neq 0$ for every root α ; moreover if $\alpha \not\equiv 0$ on t_1, i.e. $\alpha = \alpha_o + \alpha_1$ with $\alpha_1 \neq 0$ the sign of $\phi(\alpha)$ equals the sign of $\phi_1(\alpha_1)$. Thus, setting $\Phi^+ = \{\alpha \in \Phi | \phi(\alpha) > 0\}$ we have the required choice of positive roots. Let us use the following notations

$$\Phi_o = \{\alpha \in \Phi | \ \alpha|t_1 = 0\}, \quad \Phi_1 = \Phi - \Phi_o .$$

Clearly $\Phi_o = \{\alpha \in \Phi | \alpha^\sigma = \alpha\}$ while by the previous lemma σ interchanges Φ_1^+ with Φ_1^-.

Having fixed Φ^+ as in the above lemma we denote by $B \subset G$ the corresponding Borel subgroup and by B^- its opposite Borel subgroup.

1.3. It is now easy to describe the Lie algebra $\underline{h}$ in terms of the root decomposition. We have already noticed that $\sigma(\underline{g}_\alpha) = \underline{g}_{\alpha^\sigma}$.

LEMMA. If $\alpha \in \Phi_o$, σ is the identity on g_α .

PROOF. Let x_α , y_α , h_α be the standard sl_2 triple associated to α. Since $\alpha^\sigma = \alpha$ we have $\sigma(h_\alpha) = h_\alpha$. On the other hand since $\sigma(g_{\pm\alpha}) = g_{\pm\alpha}$ we have $\sigma(x_\alpha) = \pm x_\alpha$. Now if $\sigma(x_\alpha) = -x_\alpha$ we must have also $\sigma(y_\alpha) = -y_\alpha$ since $h_\alpha = [x_\alpha , y_\alpha]$. Now if we consider any element $s \in \underline{t}_1$ we have $[x_\alpha , s] = [y_\alpha , s] = 0$ since α vanishes on $\underline{t}_1$ by hypothesis. This implies, setting $t = x_1 + y_1$, that $\underline{t}_1 + Ct$ is a Toral subalgebra on which σ acts as -1. Since we can enlarge this to a maximal Toral subalgebra, we contradict the choice of T maximizing the dimension of T_1.

PROPOSITION. $\underline{h} = \underline{t}_o + \sum_{\alpha \in \Phi_o} g_\alpha + \sum_{\alpha \in \Phi_1} C(x_\alpha + \sigma(x_\alpha))$.

PROOF. Trivial from the previous lemma.

We may express a consequence of this, the so called Iwasawa decomposition: The subspace $\underline{t}_1 + \sum_{\alpha \in \Phi_1^+} Cx_\alpha$ is a complement to $\underline{h}$ and so it projects isomorphically onto the tangent space of G/H at H, in particular since Lie $B \supset t_1 + \sum_{\alpha \in \Phi_1^+} Cx_\alpha$, $BH \subset G$ is dense in G.

COROLLARY. dim G/H = dim $\underline{t}_1$ + $1/2|\Phi_1|$.

1.4. If $\Gamma \subset \Phi_+$ is the set of simple roots, let us denote $\Gamma_o = \Gamma \cap \Phi_o$, $\Gamma_1 = \Gamma \cap \Phi_1$ explicitely:

$$\Gamma_o = \{\beta_1,\ldots,\beta_k\}; \qquad \Gamma_1 = \{\alpha_1,\ldots,\alpha_j\}.$$

LEMMA. For every $\alpha_i \in \Gamma_1$ we have that α_i^σ is of the form $-\alpha_k - \Sigma n_{ij}\beta_j$ for some $\alpha_k \in \Gamma_1$ and some non negative integers n_{ij}. Moreover, $\alpha_k^\sigma = -\alpha_i - \Sigma n_{ij}\beta_j$.

PROOF. By Lemma 1.2 we know that $\alpha_i^\sigma \in \Phi^-$ hence we can write $\alpha_i^\sigma = -(\Sigma m_{ik}\alpha_k + \Sigma n_{ij}\beta_j)$ where m_{ik} , n_{ij} are non negative integers. Thus $\alpha_i = \alpha_i^{\sigma\sigma} = \sum_k m_{ik}(\sum_t m_{kt}\alpha_t) + \Sigma m_{ik}\Sigma n_{kj}\beta_j - \Sigma n_{ij}\beta_j$. Since the simple roots are a basis of the root lattice we must have in particular $\Sigma m_{ik}m_{kt} = 0$ for $t \neq i$ and $\Sigma m_{ik}m_{ki} = 1$. Since the m_{ij}'s are non negative integers it follows that only one m_{ik} is non zero and equal to 1 and the m_{ki} is also equal to 1.

Now consider the fundamental weights. Since they form a dual basis of the simple coroots we also divide them:

$$\omega_1,\ldots,\omega_j, \qquad \zeta_1,\ldots,\zeta_k \qquad \text{where:}$$

$(\omega_i , \check\beta_j) = 0$, $(\omega_i , \check\alpha_j) = \delta_j^i$ and similarly for the ζ_j's.

Since σ preserves the Killing form we have:

$$(\omega_i^\sigma , \check\beta_j^\sigma) = (\omega_i^\sigma , \check\beta_j) = 0$$

$$\delta_j^i = (\omega_j^\sigma , \check\alpha_i^\sigma) = (\omega^\sigma , \frac{2}{(\alpha_i , \alpha_i)}(-\alpha_k - \Sigma n_{ij}\beta_j))$$

$$= -(\omega_j^\sigma , \frac{2\alpha_k}{(\alpha_i , \alpha_i)}) = \frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)} (\omega_j^\sigma , \check\alpha_k)$$

We deduce that

$$\omega_i^\sigma = - \frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)} \omega_k .$$

Now ω_i^σ must be in the weight lattice so $\frac{(\alpha_k , \alpha_k)}{(\alpha_i , \alpha_i)}$ is an integer. Reversing the role of i and k we set that it must be 1 so

$$\omega_i^\sigma = -\omega_k .$$

We can summarize this by saying that we have a permutation $\tilde{\sigma}$ of order 2 in the indices 1,2,...,j such that $\omega_i^\sigma = -\omega_{\tilde{\sigma}(i)}$.

DEFINITION. A dominant weight is special if it is of the form $\Sigma n_i\omega_i$ with $n_i = n_{\tilde{\sigma}(i)}$. A special weight is regular if $n_i \neq 0$ for all i.

Thus we have that a weight λ is special iff $\lambda^\sigma = -\lambda$.

1.5.

LEMMA. Let λ be a dominant weight and let V_λ the corresponding irreducible representation of G with highest weight λ. Then if V_λ^H denotes the subspace of V_λ of H-invariant vectors dim $V_\lambda^H \leq 1$ and if $V_\lambda^H \neq 0$ λ is a special weight.

PROOF. Recall that $BH \subset G$ is dense in G so that H has a dense orbit in G/B. Also $V_\lambda \underset{G}{\cong} H^0(G/B,L)$ for a suitable line bundle L on G/B. So if $s_1,s_2 \in V_\lambda^H - \{0\}$, we have that $\frac{s_1}{s_2}$ is a meromorphic function on G/B constant on the dense H orbit, hence s_1 is a multiple of s_2 and our first claim follows.
Now assume $V_\lambda^H \neq 0$ and let $h \in V_\lambda^H - \{0\}$. Fix an highest weight vector $v_\lambda \in V_\lambda$ and let $U \subset V_\lambda$ be the unique T-stable complement to v_λ. Clearly U is B^- stable and $B^-H \subset G$ is dense in G. Then assume $h \in U$ but an the other hand B^-Hh spans V_λ a contradiction. Hence

$$h = av_\lambda + u \quad , \quad a \in \mathbb{C} - \{0\},\ u \in U$$

Since $T_0 \subset H$ and h is H invariant this implies $\lambda|T_0 = id$ hence λ is special.

1.6. If λ is any integral dominant weight and V_λ^* the corresponding irreducible representation of G with highest weight λ, we define V_λ^σ to be the space V_λ with G action twisted by σ (i.e. we set $g \circ v$ in V_λ^σ to be $\sigma(g)v$, in V_λ).

LEMMA. If λ is a special weight then V_λ^σ is isomorphic to V_λ^*.

PROOF. V_λ^* can be characterized as the irreducible representation of G having $-\lambda$ as lowest weight. Now let $v_\lambda \in V_\lambda$ be a vector of weight λ, let P be the parabolic subgroup of G fixing the line through v_λ. P is generated by the Borel subgroup B and the root subgroups relative to the negative roots $-\alpha$ for which $\langle \alpha,\lambda \rangle = 0$. Thus the parabolic subgroup P^σ, transformed of P via σ, contains the root subgroups relative to the roots $\pm\beta_i$ and also to the roots α^σ, $\alpha \in \phi_1^+$. Now $\sigma(\phi_1^+) = \phi_1^-$ hence P^σ contains the opposite Borel subgroup B^-. Clearly $v_\lambda \in V_\lambda^\sigma$ is stabilized by P^σ hence v_λ is a minimal weight vector and its weight is $-\lambda$. This proves the claim.

1.7. We have just seen that, if λ is an integral dominant special weight V_λ is isomorphic, in a σ-linear way, to V_λ^*. Under this isomorphism the highest weight vector v_λ is mapped into a lowest weight vector in V_λ^*. We normalize the mapping as follows: In V_λ the line $\mathbb{C}v_\lambda$ has a unique T-stable complement $\bar{V}_\lambda$ we define $v^\lambda \in V_\lambda^*$ by: $\langle v^\lambda, v_\lambda \rangle = 1$, $\langle v^\lambda, \bar{V}_\lambda \rangle = 0$. v^λ is easily seen to be a lowest weight vector in V_λ^*. We thus define $h: V_\lambda^* \to V_\lambda$ to be the (unique) σ-linear isomorphism such that $h(v^\lambda) = v_\lambda$.

REMARK. If $V = \oplus V_{\lambda_i}$ is a G-module, the action of G on $\mathbb{P}(V)$ factors through $\bar{G}$ if and only if the center of G acts on each V_{λ_i} with the same character. This applies in particular when V is a tensor product of irreducible G-modules.

We now analyze the stabilizer in G, $\tilde{H}$; of the line generated by h.

LEMMA. i) $\tilde{H}$ equals the normalizer of H.

ii) We have an exact sequence $H \hookrightarrow \tilde{H} \twoheadrightarrow C$, where C is the subgroup of the center of G formed by the elements expressible as $g\sigma(g^{-1})$ for some $g \in G$.

iii) The stabilizer of the line generated by h in $\bar{G}$ is the subgroup fixed by the order two automorphism induced by σ on $\bar{G}$.

PROOF. Assume ${}^gh = \alpha h$, α a scalar. Since h is σ linear, ${}^gh = ghg^{-1} = g\sigma(g^{-1})h$. Therefore $g\sigma(g^{-1})$ acts on V_λ as a scalar. Since V_λ is irreducible this implies $g\sigma(g^{-1})$ lies in the center of G. Conversely if $g\sigma(g^{-1})$ lies in the center of G, $g \in \tilde{H}$. We claim $g \in N(H)$. In fact putting $\zeta = g\sigma(g^{-1})$ we get for each $u \in H$

$$\sigma(g^{-1}ug) = \sigma(g^{-1})u\sigma(g) = \sigma(g^{-1})\zeta^{-1}u\zeta\sigma(g) = g^{-1}ug.$$

Now assume $g \in N(H)$. To see that $g \in \tilde{H}$ it is sufficient to show that $g\sigma(g^{-1})$ lies in the center of G or equivalently that it acts trivially on $\underline{g}$ = Lie G via the adjoint representation. Decompose $\underline{g} = \underline{h} \oplus \underline{g}_1$. And

consider the subgroup K in Aut($\underline{g}$) generated by adN(H) and σ. Since adN(H) is reductive and has at most index 2 in K(N(H) is clearly σ stable) also K is reductive. We claim that both $\underline{h}$ and $\underline{g}_1$ are K stable. In fact $\underline{h}$ is clearly K stable and the reductivity of K implies that it has a K-stable complement in $\underline{g}$, but the unique σ stable complement of $\underline{h}$ is $\underline{g}_1$ so $\underline{g}_1$ is also K stable.
Now notice that since $g \in N(H)$, for each $u \in H$

$$g^{-1}ug = \sigma(g^{-1})u\sigma(g)$$

so that $g\sigma(g^{-1})$ commutes with H and acts trivially on $\underline{h}$. On the other-hand, if $x \in \underline{g}_1$, we have $\mathrm{ad}g^{-1}(x) \in \underline{g}_1$, since $\underline{g}_1$ is K stable, so

$$-\mathrm{ad}g^{-1}(x) = \sigma(\mathrm{ad}g^{-1}(x)) = -\mathrm{ad}\sigma(g^{-1})(x)$$

and hence $\mathrm{ad}g\sigma(g^{-1})(x) = x$ so $g\sigma(g^{-1})$ acts trivially also on $\underline{g}_1$, and so on $\underline{g}$. This proves i).
ii) is clear from the above.
To see iii) notice that the subgroup fixing the line generated by h in $\bar{G}$ is the image in $\bar{G}$ of $\tilde{H}$. Hence if we denote by σ' the automorphism induced by σ on $\bar{G}$ it consists of the elements such that $g\sigma'(g^{-1}) = id$ which are the elements fixed by σ'.

REMARKS. a) H has finite index in $\tilde{H}$..
b) $\tilde{H}$ is the largest subgroup of G with Lie$\tilde{H} = \underline{h}$.

PROOF. a) follows from part ii) of the previous lemma and b) from the fact that H is connected (cf. [28]).

We complete v_λ to a basis $\{v_\lambda, v_1, v_2, \ldots, v_m\}$ of weight vectors and consider the dual basis $\{v^\lambda, v^1, v^2, \ldots, v^m\}$ in V_λ^*. We have $h(v^\lambda) = v_\lambda$ and, if χ_i is the weight of v_i we have $-\chi_i$ as weight of v^i and so $-\chi_i^\sigma$ as weight of $w_i = h(v^i)$. If we identify $\hom(V_\lambda^*, V_\lambda)$ with $V_\lambda \otimes V_\lambda$ we see that h is identified with the tensor

$$h = v_\lambda \otimes v_\lambda + \sum_{i=1}^{m} w_i \otimes v_i.$$

$v_\lambda \otimes v_\lambda$ has weight 2λ while $w_i \otimes v_i$ has weight $\chi_i - \chi_i^\sigma$.

The fact that h is σ-linear implies in particular that it is an H isomorphism. This in turn means that $\bar{h}$ is fixed under H.
Recall that $v_\lambda \otimes v_\lambda$ generates in $V_\lambda \otimes V_\lambda$ the irreducible module $V_{2\lambda}$.
Now order $\alpha_1, \ldots, \alpha_j$ so that $\alpha_s - \alpha_s^\sigma$ are mutually distinct for $s \leq \ell$ (and of course by 1.4 if $j > \ell$, for each $i > \ell$ there is an index $s \leq \ell$ such that $\alpha_s - \alpha_s^\sigma = \alpha_i - \alpha_i^\sigma$). Call $\bar{\alpha}_s = \frac{1}{2}(\alpha_s - \alpha_s^\sigma)$ $s \leq \ell$ the restricted simple roots.

PROPOSITION. i) If λ is a special weight then $V_{2\lambda}$ contains a non zero element h' fixed under H.
ii) h' is unique up to scalar multiples and can be normalized to be

$$h' = v_{2\lambda} + \sum z_i$$

with $v_{2\lambda}$ a highest weight vector of $V_{2\lambda}$ and the z_i's weight vectors having distinct weights whose weight is of the form $2(\lambda - \sum_{s=1}^{\ell} n_s \bar{\alpha}_s)$, n_i non negative integers.
iii) if λ is a regular special weight then we can assume that the vectors $z_1, \ldots, z_\ell$ have weight $2(\lambda - \bar{\alpha}_1), \ldots, 2(\lambda - \bar{\alpha}_\ell)$.

PROOF. If we put h' equal to the image of $\bar{h}$ under the unique G-equivariant projection $V_\lambda \otimes V_\lambda \to V_{2\lambda}$, i) ii) follow from the expression of $\bar{h}$ as a linear combination of weight vectors given above. To see iii) assume λ (and hence 2λ) is a regular special weight. Since h' is fixed under H, $xh' = 0$ for any $x \in \underline{h}$ = LieH. In particular if we let $\bar{\alpha}_s$ be a simple restricted root and $\alpha_s \in \Gamma_1$ be such that $\bar{\alpha}_s = \frac{1}{2}(\alpha_s - \alpha_s^\sigma)$ we have (cf. 1.3)

$$(x_{-\alpha_s} + \sigma(x_{-\alpha_s}))h' = 0, \quad x_{-\alpha_s} \in g_{-\alpha_s}.$$

But

$$(x_{-\alpha_s} + \sigma(x_{-\alpha_s}))v_{2\lambda} = x_{-\alpha_s} v_{2\lambda}$$

since $\sigma(x_{-\alpha_s}) \in g_{-\alpha_s^\sigma}$ and $-\alpha_s^\sigma \in \Phi_1^+$. Also by the regularity of 2λ $x_{-\alpha_s} v_\lambda$ is a non zero weight vector of weight $2\lambda - \alpha_s$. It follows that for some z_i, $\sigma(x_{-\alpha_s})z_i = -x_{-\alpha_s} v_{2\lambda}$ so that z_i has weight $2(\lambda - \bar{\alpha}_s)$ proving the claim.

The analysis just performed does not exclude that V_λ itself may contain a non zero H-fixed vector h_λ. In this case we have seen that we can normalize h_λ : $h_\lambda = v_\lambda + \sum u_i^\lambda$, u_i^1 lower weight vectors. It follows that $h_\lambda \otimes h_\lambda$ must project to h in $V_{2\lambda}$ (by uniqueness of h).

Now the dominant λ's for which $\dim V_\lambda^H = 1$ have been determined completely [9],[24], the result is as follows: Let us indicate Λ^1 such set.

Consider the Killing form restricted to $\underline{t}_1$ and thus to $\underline{t}_1^*$. We look at the restriction of Φ_1 to $\underline{t}_1$, if $\alpha \in \Phi$, let us indicate $\bar{\alpha}$ the restriction of α to $\underline{t}_1$.

If $\mu \in \underline{t}_1^*$ let us indicate by $\tilde{\mu}$ its extension to $\underline{t}$ by setting it 0 to $\underline{t}_0$.
Then the theorem in [9] is:
Consider the set of $\mu \in \underline{t}_1^*$ such that

$$\frac{(\mu,\bar\alpha)}{(\bar\alpha,\bar\alpha)} \quad \text{is a positive integer for all } \alpha \in \phi$$

Then the set of weights $\tilde\mu$ of $\underline{t}$ so obtained is exactly the set Λ^1 of λ for which dim $V_\lambda^H = 1$. One can understand this theorem in a more precise way. If $\alpha \in \phi$, then $\tilde\alpha$ is exactly $\frac{1}{2}(\alpha - \alpha^\sigma)$, and $(\bar\alpha;\bar\alpha) = (\tilde\alpha,\tilde\alpha)$. Now also a weight ω is of the form $\tilde\mu$ if and only if $\omega = \frac{1}{2}(\omega - \omega^\sigma)$. For such weights of course $(\omega,\beta_j) = 0$. Thus we see immediately that Λ^1 is contained in the positive lattice generated by the weights ω_i if $\sigma(i) = i$ and $\omega_i - \omega_{\tilde\sigma(i)}$ if $\tilde\sigma(i) \neq i$.

To understand exactly the nature of Λ^1 we must see if

$$\frac{(\omega_i,\bar\alpha)}{(\bar\alpha,\bar\alpha)} \qquad \left(\text{resp. } \frac{(\omega_i - \omega_{\tilde\sigma(i)},\bar\alpha)}{(\bar\alpha,\bar\alpha)}\right)$$

is an integer.

Since in any case for such special weights λ we have $2\lambda \in \Lambda^1$ one knows at least that these numbers are half integers. It follows in any case that Λ^1 is the positive lattice generated by the previous weights or their doubles. i.e.

$$\Lambda^1 = \sum_{i=1}^{\ell} n_i\mu_i, \quad n_i \geq 0 \quad \text{and} \quad \mu_i = \omega_i \quad \text{or}$$

$2\,\omega_i$ (resp. $\omega_i - \omega_{\tilde\sigma(i)}$ or $2(\omega_i - \omega_{\tilde\sigma(i)})$). Recall that $\ell = \text{rk}\,\Lambda^1$ is also the rank of the symmetric space.

2. THE BASIC CONSTRUCTION

2.1. We consider now a regular special weight λ and all the objects of the previous paragraph V_λ, $h' \in V_{2\lambda}$. Let now $\mathbb{P}_{2\lambda} = \mathbb{P}(V_{2\lambda})$ be the projective space of lines in $V_{2\lambda}$ and $\tilde h \in \mathbb{P}_{2\lambda}$ be the class of h'. The basic object of our nalysis is the orbit $G \cdot \tilde h$ of $\tilde h$ in $\mathbb{P}_{2\lambda}$ and its closure $\bar X = \overline{G \cdot \tilde h}$. By construction $\bar X$ is a G-equivariant compactification of the homogeneous space $G \cdot \tilde h$, furthermore the stabilizer $\hat H$ of $\tilde h$ is a group containing the subgroup H.

We will analyze in detail $\tilde H$ and in particular will see that H has finite index in $\tilde H$. For the moment we concentrate our attention to $\bar X$. Since $\bar X$ is closed in $\mathbb{P}_{2\lambda}$ and G stable it contains the unique closed orbit of G acting on $\mathbb{P}_{2\lambda}$, i.e. the orbit of the highest weight vector $v_\lambda \otimes v_\lambda$. Now the following general lemma is of trivial verification:

LEMMA: If X is a G variety with a unique closed orbit Y and V is an open set in X with $Y \cap V \neq \phi$ then $X = \bigcup_{g\in G} gV$.

The use of this lemma for us is in the fact that it allows us to study the singularities of X locally in V.

2.2. Let λ be a regular special weight. Consider a G module $W \simeq V_{2\lambda} \oplus \sum V_{\mu_i}$ with $\mu_i = 2\lambda - \sum n_i 2\bar\alpha_i$ some $n_i > 0$. Let $h \in V$ be an H invariant with component h' in $V_{2\lambda}$. Decompose $V_{2\lambda} = Cv_{2\lambda} \oplus \tilde V_{2\lambda}$ in a T stable way and consider the open affine set $A = v_{2\lambda} \oplus \tilde V_{2\lambda} \oplus \sum v_{\mu_i} \subseteq \mathbb{P}(W)$. Notice that $h \in A$ and A is B^- stable.

LEMMA: The closure in A of the T^1 orbit T^1h is isomorphic to ℓ dimensional affine space $\mathbb{A}^\ell$. The natural morphism $T^1 \to T^1h \hookrightarrow \mathbb{A}^\ell$ has coordinates $t \to (t^{-2\bar\alpha_1}, t^{-2\bar\alpha_2}, \ldots, t^{-2\bar\alpha_\ell})$. T^1h is identified with the open set of $\mathbb{A}^\ell$ where all coordinates are non zero.

PROOF: By prop. 1.7 we can write $h = v_{2\lambda} + \sum_i z_i'$ with z_i' weight vectors of weights $\chi_i = 2\lambda - \sum m_j^{(i)} 2\bar\alpha_j$ (some $m_j > 0$) and $z_1', \ldots, z_\ell'$ of weights $2\lambda - 2\bar\alpha_1, \ldots, 2\lambda - 2\bar\alpha_\ell$. Let us apply an element $t \in T'$ to h we get $th = t^{2\lambda}v_{2\lambda} + \sum t^{\chi_i} z_i'$ which, in affine coordinates, is

$$v_{2\lambda} + \sum t^{\chi_i - 2\lambda} z_i'.$$

From the previous formula $\chi_i - 2\lambda = \sum_j m_j^{(i)}(-2\bar\alpha_j)$, this means that the coordinates of th are monomials in the first ℓ coordinates.

This means that T^1 maps to a closed subvariety of A, isomorphic to affine space $\mathbb{A}^\ell$, via the coordinates $(t^{-2\bar\alpha_1}, \ldots, t^{-2\bar\alpha_\ell})$. Since the restricted simple roots are linearly independent the orbit T^1h is the open dense subset of $\mathbb{A}^\ell$ consisting of the elements with non zero coordinates.

REMARK. The stabilizer of h in T^1 is the finite subgroup of the element: $t \in T^1$ with $t^{2\bar\alpha_i} = 1$.

2.3. Let us go back to $\bar X \subseteq P_{2\lambda}$. Consider the open affine set $A = v_{2\lambda} \oplus \tilde V_{2\lambda} \subseteq P_{2\lambda}$ and set $V = A \cap \bar X$. Remark that V is B^- stable, it contains $\tilde h$ and so also $\mathbb{A}^\ell$, the closure of $T^1\tilde h$ in A, hence $v_{2\lambda} \in V$ and therefore V has a non empty intersection with the unique closed orbit or G in $P_{2\lambda}$.

Let U be the unipotent group generated by the root subgroups X_α, $\alpha \in \phi_1^-$. Since U acts on V we have a well defined map $\phi: U \times \mathbb{A}^\ell \to V$ by the formula $\phi(u,x) = u\cdot x$.

PROPOSITION: $\phi : U \times \mathbb{A}^\ell \to V$ is an isomorphism.

PROOF. We first will construct a map $\psi: V \to U$ such that $\psi\phi(u,x) = u$, and prove that Im ϕ is dense in V. From this the claim follows; in fact consider the map $\zeta: V \to V$ given by $\zeta(v) = \psi(v)^{-1}v$, clearly $\zeta\phi(u,x) = x$ hence ζ maps V in $\mathbb{A}^\ell$ and setting $\phi': V \to U \times \mathbb{A}^\ell$ by $\phi'(v) = (\psi(v),\zeta(v))$ we have $\phi'\cdot\phi = 1_{U\times\mathbb{A}^\ell}$. Since $\phi(U \times \mathbb{A}^\ell)$ is dense in V and $\phi\cdot\phi'$ is the identity we also have $\phi\cdot\phi' = 1_V$.

2.4. From now on we make the necessary steps for the construction of ψ. Since 2λ is special we have, by our considerations of 1.6, that $V_{2\lambda}$ is isomorphic to $V^*_{2\lambda}$ in a σ-linear way. This isomorphism defines a non degenerate bilinear form $\langle\,,\rangle$ on $V_{2\lambda}$ which is symmetric and satisfies the following properties:

$$\langle gu,v\rangle = \langle u,\sigma(g^{-1})v\rangle \quad \text{for each } g \in G,\ u,v \in V_{2\lambda}$$

$$\langle xu,v\rangle = -\langle u,\sigma(x)v\rangle \quad \text{for each } x \in \underline{g},\ u,v \in V_{2\lambda}$$

Remark that the tangent space τ in $v_{2\lambda}$ to the orbit $U\cdot v_{2\lambda}$ has as basis the elements $x_{-\alpha}v_{2\lambda}$, $\alpha \in \phi_1^+$ (since the opposite unipotent group of U is the unipotent radical of the parabolic subgroup P stabilizing the line through $v_{2\lambda}$). Let τ^o be the subspace generated by τ and $v_{2\lambda}$.

LEMMA: i) The form $\langle\,,\rangle$ restricted to τ^o is non degenerate.
ii) τ^o is stable under P.
iii) The orthogonal $\tau^{o\perp}$ (relative to the given form) is stable under $\sigma(P)$.

PROOF: i) First of all remark that if $v_1,v_2 \in V_{2\lambda}$ are weight vectors of weights χ_1,χ_2 respectively and $\langle v_1,v_2\rangle \neq 0$ we have, for $t \in T$, $t^{\chi_1}\langle v_1,v_2\rangle = \langle tv_1,v_2\rangle = \langle v_1,\sigma(t^{-1})v_2\rangle = t^{-\chi_2^\sigma}\langle v_1,v_2\rangle$ and so $\chi_1 = -\chi_2^\sigma$. This implies that $v_{2\lambda}$ is orthogonal to $\tilde{V}_{2\lambda}$ and $\langle v_{2\lambda},v_{2\lambda}\rangle \neq 0$.
It remains to verify that on τ the form is non degenerate. Using our previous remark $\langle x_{-\alpha}v_{2\lambda},x_{-\beta}v_{2\lambda}\rangle = 0$ unless $\beta = -\alpha^\sigma$. In this case $\langle x_{-\beta}\sigma v_{2\lambda},x_{-\beta}v_{2\lambda}\rangle = -c\langle v_{2\lambda},x_\beta x_{-\beta}v_{2\lambda}\rangle$ $(c \neq 0)$ and $\langle v_{2\lambda},x_\beta x_{-\beta}v_{2\lambda}\rangle =$ $= \langle v_{2\lambda},[x_\beta,x_{-\beta}]v_{2\lambda}\rangle$ since $x_\beta v_{2\lambda} = 0$ this is $(2\lambda,\beta)\langle v_{2\lambda},v_{2\lambda}\rangle \neq 0$.
Since the map $\alpha \to -\alpha^\sigma$ is an involution of ϕ_1^+ the first claim follows.
ii) It is sufficient to show that τ^o is stable under the action of the Lie algebra of P. Since τ^o is stable under the torus T it is enough to show the stability of τ^o with respect to the elements x_α with $\alpha \in \phi_o \cup \phi_1^+$. Now $x_\alpha x_{-\beta}v_{2\lambda} = [x_\alpha,x_{-\beta}]v_{2\lambda} + x_{-\beta}x_\alpha v_{2\lambda}$, if $\alpha \in \phi_o \cup \phi_1^+$ we have $x_\alpha v_{2\lambda} = 0$.
iii) This is clear from the properties of the form.

2.5.

LEMMA. $\mathbb{A}^\ell \subseteq v_{2\lambda} + \tau^{o\perp}$.

PROOF. We must show that, if $h' = v_{2\lambda} + \sum z_i$, each $z_i \in \tau^{o\perp}$. The weight of z_i is $\chi_i = 2\lambda - \sum n_j^{(i)}2\bar{\alpha}_j$ so the only case to verify is when $-\sum n_j^{(i)}2\bar{\alpha}_j = -\beta$ for some $\beta \in \phi_1^+$. Suppose this happens for z_{i_o}, since h' is H stable we have $(x_\beta + \sigma(x_\beta))h' = 0$; but $(x_\beta + \sigma(x_\beta))h' = x_\beta z_{i_o} +$ terms of weight different from 2λ, thus $x_\beta z_{i_o} = 0$. By the same weight considerations the only possible non zero scalar product between z_{i_o} and the elements of the basis of τ^o is the one with $x_{-\beta}v_{2\lambda}$, for this we have $\langle x_{-\beta}v_{2\lambda},z_{i_o}\rangle = -\langle v_{2\lambda},\sigma(x_{-\beta})z_{i_o}\rangle = 0$, $(\sigma(x_{-\beta}) = cx_\beta$ some c).

2.6. Now we consider the projection π of $V_{2\lambda}$ onto $V_{2\lambda}/\tau^{o\perp}$, since $U \subset \sigma(P)$ we have a U action on $V_{2\lambda}/\tau^{o\perp}$ and the projection is equivariant. Let $K = \pi(v_{2\lambda} + \tilde{V}_{2\lambda})$, K is an affine hyperplane in $V_{2\lambda}/\tau^{o\perp}$ and it is U stable.

LEMMA. The map $j: U \to K$ defined by $j(u) = \pi(uv_{2\lambda})$ is a U equivariant isomorphism.

PROOF. From 2.4 we know that τ is the tangent space of $Uv_{2\lambda}$ in $v_{2\lambda}$. This implies that j is smooth at 1. Since j is U equivariant it is everywhere smooth. Now U has no finite subgroups and $\dim U = \dim K$ so j is an open immersion. It is a well known fact that an open immersion j of affine space $\mathbb{A}^n$ into another affine space $\bar{\mathbb{A}}^n$ of the same dimension is necessarily an isomorphism, we recall the proof: It the complement of $j(\mathbb{A}^n)$ is non emty it is a divisor which has an equation f, this is a unit a $\mathbb{A}^n$ and hence a constant, giving a contradiction.
We can now construct ψ as required in 2.3, setting $\psi(v) = j^{-1}(\pi(v))$ for any $v \in V$, the fact that $\psi\phi(u,x) = u$ follows from the U equivariance of π and j and lemma 2.5.

2.7.

LEMMA. The image of ϕ is dense in V.

PROOF. The tangent space to $\mathbb{A}^\ell$ in $v_{2\lambda}$ is orthogonal to τ (cf. 2.5). This implies that the differential of ϕ in the point (1,0) is injective and so $\dim(\overline{\mathrm{Im}\phi}) = \dim(U\times\mathbb{A}^\ell)$; now $\dim V = \dim\underline{X} \leq \dim G/H = \dim(U\times\mathbb{A}^\ell)$. Since V is irreducible we get that $V = \overline{\mathrm{Im}\phi}$.

PROPOSITION. The stabilizer of $\tilde{h}$ is $\tilde{H}$.

PROOF. We have shown in the previous lemma that $\dim X = \dim G/H$ hence the subgroup H has finite index in the stabilizer of $\tilde{h}$. From 1.7 the proposition follows.

2.8. Using proposition 2.3 we identify V with the affine space $U \times \mathbb{A}^{\ell}$.

PROPOSITION. The intersection between the orbit $G\tilde{h}$ and $U \times \mathbb{A}^{\ell}$ is the open set where the last ℓ coordinates are non zero.

PROOF. We go back to $h \in \hom(V_{\lambda}^{*},V_{\lambda}) \simeq V_{\lambda} \otimes V_{\lambda}$ (cf. 1.7) and proceed as in 2.1, 2.2. Let $h^{\#}$ be the class of h in $\mathbb{P}(\hom(V_{\lambda}^{*},V_{\lambda})) = \mathbb{P}(V_{\lambda} \otimes V_{\lambda})$ and $\underline{X}^{\#} = \overline{G \cdot h^{\#}}$. Setting $V_{\lambda} \otimes V_{\lambda} = V_{2\lambda} \oplus Z$, the decomposition in G submodules, we consider the affine space $A^{\#} = v_{2\lambda} + \tilde{V}_2 \oplus Z$ and the G equivariant projection $\rho: \mathbb{P}(V_{\lambda} \otimes V_{\lambda}) \to \mathbb{P}(V_{2\lambda})$ from $\mathbb{P}(Z)$, ρ is defined in the open set $\mathbb{P}(V_{\lambda} \otimes V_{\lambda}) - \mathbb{P}(Z)$, hence in particular in $V^{\#} = \underline{X}^{\#} \cap A^{\#}$.

From the analysis of 2.2 the closure in $A^{\#}$ of the orbit $T^{1}h^{\#}$ projects under ρ isomorphically onto $\mathbb{A}^{\ell}$ hence the isomorphism $\phi: U \times \mathbb{A}^{\ell} \to V$ factors through $\phi: U \times \mathbb{A}^{\ell} \xrightarrow{\phi^{\#}} V^{\#} \xrightarrow{\rho} V$. We know that $\dim V^{\#} = \dim X^{\#} = \dim G/\tilde{H}$ (cf. 1.7) so $\mathrm{Im}\,\phi^{\#}$ is dense in $V^{\#}$ and as in 2.3 this implies that $\phi^{\#}$ is an isomorphism. We now have that the union of the translates of $V^{\#}$ under G is an open dense subset in $X^{\#}$ isomorphic, under ρ, to $\bar{X}$; since $\bar{X}$ is complete this open set must be $\underline{X}^{\#}$. We can now prove the proposition working with $V^{\#}$, $\underline{X}^{\#}$ and $Gh^{\#}$. The points in $U \times \mathbb{A}^{\ell}$ where the last ℓ coordinates are non zero are in the B^{-} orbit of $h^{\#}$ hence in $Gh^{\#}$, we show now that the remaining points cannot be in $Gh^{\#}$. In order to do this we interpret such points as maps from V_{λ}^{*} to V_{λ} and show that an element of $\mathbb{A}^{\ell}$ with a zero coordinate is not of maximal rank, this is clear from the analysis of 1.7. Since every point in $V^{\#}$ is in the U orbit of a point in $\mathbb{A}^{\ell}$ the proposition follows.

3. THE MINIMAL COMPACTIFICATION

3.1. We can now completely describe the structure of the variety $\bar{X}$.

THEOREM.

i) $\bar{X}$ is smooth.

ii) $\bar{X} - G \cdot \tilde{h}$ is a union of ℓ smooth hypersurfaces S_i which cross transversely.

iii) The G orbits of $\bar{X}$ correspond to the subsets of the indeces $1,2,\ldots,\ell$ so that the orbit closures are the intersections $S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_k}$.

iv) The unique closed orbit $Y \simeq G/P$ is $\bigcap_{i=1}^{\ell} S_i$.

PROOF. We have seen that the complement of $G \cdot \tilde{h} \cap V$ in V is the union of ℓ hypersurfaces which are in fact coordinate hyperplanes, since $V \simeq U \times \mathbb{A}^{\ell}$ and the ℓ hypersurfaces Σ_i are given by the equations $x_i = 0$ for the last ℓ coordinates. Furthermore, the description of the torus action of T_1 on $\mathbb{A}^{\ell}$ shows that, two points in V are in the same $U \times T_1$

orbit if and only if they lie in the same set of hyperplanes Σ_i. Now we claim that the hypersurfaces S_i are just the closure of the Σ_i in $\bar{X}$. In fact, let S_i be any irreducible component of $S - G \cdot \tilde{h}$, necessarily S_i is G stable, since G is connected. Hence, $S_i \supsetneq Y$ (the unique closed orbit) and $S_i \cap V$ is thus a component of $V - G \cdot \tilde{h}$. Hence, $S_i \cap V = \Sigma_i$ (up to reordering the indeces). Hence, $S_i = \bar{\Sigma}_i$ and conversely by the same argument, $\bar{\Sigma}_i$ is an irreducible component of $X - G \cdot \tilde{h}$, hence, it is G-stable.

To finish it is only necessary to remark that, since any point is G-conjugate to a point in V, the statement that two points in $\bar{X}$ are in the same orbit if and only if they are contained in the same S_i's follows from the similar statement relative to $U \times T_1$ in V.

3.2. Summarizing, we have found ℓ hypersurfaces S_i which are smooth. The orbits are just

$$O_{i_1,\ldots,i_k} = S_{i_1} \cap \ldots \cap S_{i_k} - \bigcup_{i \neq i_1,\ldots,i_k} S_{i_1} \cap \ldots \cap S_{i_k} \cap S_i$$

and $\bar{O}_{i_1,\ldots,i_k} = S_{i_1} \cap \ldots \cap S_{i_k}$ is smooth.

These are the only irreducible, closed G-stable subsets of $\bar{X}$. Their inclusion relations are, therefore, opposite to those of the faces of the simplex on the indeces $1,2,\ldots,\ell$. The statement iv) is then clear.

3.3. We have just seen that, given a regular special weight λ we can describe the structure of the variety $\bar{X} = \overline{G\,\tilde{h}} \subset \mathbb{P}(V_{2\lambda})$. Assume now that V_{λ} itself contains a non zero H-invariant line generated by h' and consider $\bar{X}' = \overline{G \cdot \tilde{h}'} \subset \mathbb{P}(V_{\lambda})$.

PROPOSITION. There is a natural G-isomorphism $\psi: \bar{X}' \to \bar{X}$.

PROOF. Let us consider the map $\phi: V_{\lambda} \to V_{2\lambda}$ which is the composition of the map $f: V_{\lambda} \to V_{\lambda} \otimes V_{\lambda}$ defined by $f(v) = v \otimes v$ and of the G-equivariant projection $\pi: V_{\lambda} \otimes V_{\lambda} \to V_{2\lambda}$. Clearly ϕ is G-equivariant and we can normalize h' so that $\phi(h') = \tilde{h}$. If we identify V_{λ} (resp. $V_{2\lambda}$) with $H^{0}(G/B,L_{\lambda})$ (resp. $H^{0}(G/B,L_{2\lambda})$ (where L_{μ} is the line bundle relative to the dominant weight μ), we see that ϕ is the map taking a section into its square. Since G/H is irreducible, we then have that ϕ induces an embedding $\bar{\phi}: \mathbb{P}(V_{\lambda}) \to \mathbb{P}(V_{2\lambda})$ which is G-equivariant (and an isomorphism of $\mathbb{P}(V_{\lambda})$ onto its image). Clearly $\bar{X}$ is contained in the image of $\bar{\phi}$ and is the image of $\bar{X}'$. Thus $\bar{\phi}$ induces the required isomorphism ψ.

3.4. We should remark that in the special case of a group G, considered as symmetric variety over $G \times G$, one can more simply describe the construction ad follows. If λ is a regular dominant weight of G and V_λ the corresponding irreducible representation, we consider $\mathrm{End}(V_\lambda) = V_\lambda \otimes V_\lambda^*$ as $G \times G$ module. G is then thought as the orbit of the identity $1 \in \mathrm{End}(V_\lambda)$ and the compactification $X = \overline{G \cdot 1}$ can thus be thought as "degenerate" projective transformation of the flag variety. We will refer to this case as the "compactification of $\bar{G}$".

4. INDEPENDENCE ON λ

4.1. A priori the construction performed in §2 depends on the regular weight λ, we want to show now a different construction of $\bar{X}$ which shows its independence on λ. Consider again the permutation $\tilde{\sigma}$ considered in 1.3. Each orbit of $\tilde{\sigma}$ consists of either one or two indices. Indexing the orbit by the indeces $\{1,\dots,\ell\}$, for each such index j we let λ be the sum of the fundamental weights (one or two) in the corresponding orbit. Thus a special weight is just a positive integral combination $\sum n_j \cdot \lambda_j$ while a regular one has the condition $n_j \neq 0$ for all j.

For each j we have $V_{\lambda_j} \simeq V_{\lambda_j}^*$ and a corresponding element $h_j \in V_{2\lambda_j}$. Consider then $\bar{h}_j \in \mathbb{P}(V_{2\lambda_j})$ and $\bar{h}' = (h_1,\dots,h_\ell) \in \Pi\, \mathbb{P}(V_{2\lambda_j})$. We claim that $\bar{X}$ is isomorphic to $\overline{G \cdot \bar{h}'} \subseteq \Pi\, \mathbb{P}(V_{2\lambda_j})$. In fact, consider $\lambda = \sum n_j \lambda \lambda_j$ and $\bigotimes_j V_{\lambda_j}^{\otimes n_j} = Q$. Clearly $Q = V_\lambda \oplus Q'$ with Q' a sum of representations with lower highest weights. The element

$$\otimes h_j^{\otimes n_j} : \bigotimes_j V_{\lambda_j}^{\otimes n_j} \to \bigotimes_j V_{\lambda_j}^{*\otimes n_j}$$

and in particular it maps V_λ in V_λ^* and by the uniqueness of h it coincides with h on V_λ. Now we have clearly a mapping $\pi\, \mathbb{P}(V_{2\lambda_j}) \to \mathbb{P}(\otimes V_{2\lambda_j}^{\otimes n_j})$ sending $\bar{h}'$ to $\otimes h_j^{\otimes n_j}$ and so $\overline{G \cdot \bar{h}'}$ is identical to the closure of the orbit of $\otimes h_j^{\otimes n_j}$. Let $\bar{X}'$ be $G \cdot \otimes h_j^{\otimes n_j} \subseteq \mathbb{P}(\bigotimes_j V_{2\lambda_j}^{\otimes n_j})$. We wish to project $\bar{X}'$ to X proving that they are isomorphic. In fact, we prove a more general statement which will be used later. Let us give a regular special weight λ and a representation W, with a line $\mathbb{C}h_W$ fixed under H, such that its T_1 weights are all of the form $\lambda - \sum n_i 2\bar{\alpha}_i$.

Suppose $h_\lambda \in V_\lambda$ is an H-invariant non zero vector and set $h = h_\lambda + h_W \in V_\lambda \oplus W$ and $\bar{X}' = \overline{G\tilde{h}} \subseteq \mathbb{P}(V_\lambda \oplus W)$. If we project $\mathbb{P}(V_\lambda \oplus W)$ to $\mathbb{P}(V_\lambda)$ from $\mathbb{P}(W)$ we have

LEMMA. The projection is defined on $\bar{X}'$ and establishes an isomorphism between $\bar{X}'$ and $\bar{X} = \overline{G\tilde{h}_\lambda}$.

PROOF. We can assume $W = \oplus V_i$, each V_i irreducible and containing a H fixed line $\mathbb{C}h_i$ so that the projection $\Pi_i : W \to V_i$ with kernel $\bigoplus_{j \neq i} V_i$ has the property $\Pi_i(h_W) = h_i$.

By reasoning as in 3.3 we can double all weights and assume $\lambda = 2\lambda'$ and V_i has weight $2\mu_i$. In this situation we can define in $\bar{X}'$ the affine set V' as in 2.2 and carry out the same analysis verbatim due to the structure of the weights of h_W. Then we see that under the given map $\mathring{X}' = \bigcup_{g \in G} V'^g$ in $\bar{X}'$ projects isomorphically onto $\bar{X}$. Since $\bar{X}$ is complete, it follows that $\mathring{X}'$ is also complete and hence $\mathring{X}' = \bar{X}'$ as desired.

5. THE STABLE SUBVARIETIES

5.1. We have seen that in $\bar{X}$ the only G stable subvarieties are of the form $W_{i_1,\dots,i_k} = S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_k}$ for a subset of the indices $1,2,\dots,\ell$. We wish now to describe geometrically such a subvariety. Let us then consider the weights λ_j, $j = 1,2,\dots,\ell$ defined in 4.1 and the two weights $\lambda_1 = \lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}$ and $\lambda_2 = \lambda_{j_1} + \dots + \lambda_{i_{\ell-k}}$ where $j_1,\dots,j_{\ell-k}$ are the complement of $i_1,i_2,\dots,i_k$ in $1,2,\dots,\ell$. We can, as before, consider $\bar{X}$ embedded in $\mathbb{P}(V_{2\lambda_1}) \times \mathbb{P}(V_{2\lambda_2}) \subseteq \mathbb{P}(V_{2\lambda_1} \otimes V_{2\lambda_2})$ and we can project $\bar{X}$ to $\mathbb{P}(V_{2\lambda_1})$. Let us call Π_1 this projection which is clearly G equivariant and maps onto the closure of the orbit $\bar{X}_1 = \overline{G \cdot \tilde{h}_{2\lambda_1}}$.

LEMMA. $\Pi_1(W_{i_1,\dots,i_k})$ equals the unique closed orbit in $\bar{X}_1$ (i.e. G/P_1, P_1 the parabolic, stabilizing the line through a highest weight vector in $V_{2\lambda_1}$).

PROOF: We may analyze the projection locally in V and in fact, since $V = U \cdot A^\ell$, it is enough to study $\Pi_1(A^\ell \cap W_{i_1,\dots,i_k}) = \Pi_1(A^\ell_{i_1,\dots,i_k})$. We know that the intersection $A^\ell \cap W_{i_1,\dots,i_k}$ is that part $A_{i_1,\dots,i_k}$ of A^ℓ where the coordinates x_i (corresponding to $t^{-2\alpha_i}$) vanish, for $i = i_1,i_2,\dots,i_k$. The weights of the representation $V_{2\lambda_1}$, different from the highest weight, are of all of the form $\psi = 2\lambda_1 - \sum n_i\alpha_i - \sum_i \beta_i$ where at least one of the coordinates n_i relative to the indices i, for which $(\alpha_i,\lambda_1) \neq 0$, is non negative.

If we consider the projection of the subspace $A^\ell = R_1 = \overline{T_1 \tilde{h}_{2\lambda}}$, this can be analyzed as follows. We have the orbit $T_1 \cdot \tilde{h}_{2\lambda_1}$ and its closure R_1' and R_1 maps to R_1'. In coordinates we know that the T_1 weights appearing in $L_{2\lambda_1}$ are of type $2\lambda_1 - \sum n_i 2\alpha_i$ and then the corresponding mapping expresses such coordinates as $\Pi\, x_i^{n_i}$, but we know that some $n_i > 0$ for one the indices $i = i_1, i_2, \dots, i_k$. Thus we deduce that $\Pi_1(A^\ell \cap W_{i_1,\dots,i_k})$ is just the point $\overline{v_{2\lambda_1} \otimes v_{2\lambda_1}}$. This proves the lemma.

5.2. We have thus established a G equivariant mapping

$$\Pi_1 : W_{i_1,\dots,i_k} \to G \cdot \overline{v_{2\lambda_1} \otimes v_{2\lambda_1}}.$$

This last variety is of the form $G/P_{i_1,\dots,i_k}$ for the parabolic fixing $\overline{v_{2\lambda_1} \otimes v_{2\lambda_1}}$.
Since the map is G equivariant, it is a fibration. We want to study a typical fiber. Let us study $\Pi_1^{-1}(\overline{v_{2\lambda_1} \otimes v_{2\lambda_1}}) = \bar{X}_1$.

Since Π_1 is a smooth morphism $\bar{X}_1$ is smooth and is the closure of the fiber of Π_1 restricted to the open orbit in $W_{i_1\dots i_k}$; this is irreducible since P is connected. We start to study $\bar{X}_1$ locally always in the open set V. A point (γ, a) in $U_\Gamma \times A_{i_1,\dots,i_k}$ is in the fiber $\bar{X}_1$ if and only if $\gamma \cdot \overline{v_{2\lambda_1} \otimes v_{2\lambda_1}} = \overline{v_{2\lambda_1} \otimes v_{2\lambda_1}}$, i.e. if and only if $\gamma \in P_{i_1,\dots,i_k}$. Now $U \cap P_{i_1,\dots,i_k}$ is exactly the unipotent subgroup generate by the root subgroup of the roots $-\alpha_i$ where $\alpha_i \in \Gamma_1$ and also α_i is a root of the Levi subgroup of $P_{i_1,\dots,i_k}$. The semisimple part of the Levi subgroup of $P_{i_1,\dots i_k}$ is relative to the root system generated by the roots β_j and the roots α_k's for which $(\alpha_k, \lambda_1) = 0$. Clearly such a subgroup $L_{i_1,\dots,i_k}$ is σ stable. Moreover, if we consider $A_{i_1,\dots,i_k} \subseteq \mathbb{P}(V_{2\lambda_2})$, we can analyze it as follows:
$h_{2\lambda_2} = v_{2\lambda_2} \otimes v_{2\lambda_2} + \sum z_i'$ where z_i' has T_1 weight $2\lambda_2 - \sum m_j 2\alpha_j$. We can split $h_{2\lambda_2}$ as $h_{2\lambda_2} = h'_{2\lambda_2} + a'$ where a' is the sum of all terms of weight $2\lambda_2 - \sum m_j 2\alpha_j$ with $m_j \neq 0$ for some $j \in \{i_1, i_2, \dots, i_k\}$. Consider any element $t \in T_1$ such that t commutes with the Levi subgroup $L_{i_1,\dots,i_k}$. Consider $H_{i_1,\dots,i_k} = L_{i_1,\dots,i_k} \cap H$, we have if $g \in H_{i_1,\dots,i_k}$, $t^{-1} \cdot gt = g$ and so $t\, h_{2\lambda_2} = g \cdot t \cdot h_{2\lambda_2}$. Hence,

$$h'_{2\lambda_2} + t \cdot a' = g \cdot h'_{2\lambda_2} + g \cdot t \cdot a'.$$

We deduce that $h'_{2\lambda_2} = g \cdot h'_{2\lambda_2}$ so $h'_{2\lambda_2}$ is $H_{i_1,\dots,i_k}$ invariant. Moreover, we see that $A_{i_1,\dots,i_k}$ can be considered as the closure of the action of the Torus $(T_1)_{i_1\dots i_k}$ on $\tilde{h}'_{2\lambda_2}$.

Thus, we deduce that the fibre we are studing is in fact the closure of the orbit of the semisimple part of the Levi subgroup acting on $\tilde{h}'_{2\lambda_2}$. Since it is easily verified that $(T_1)_{i_1\dots i_k}$ is a maximal

anisotropic in $L_{i_1\dots i_k}$ and λ_2 restricted to $T \cap L_{i_1\dots i_k}$ is a regular special weight we can apply the general remarks and lemma 5.1, and see that X_1 is isomorphic to the minimal compactification of the corresponding symmetric algebraic variety $\bar{L}_{i_1\dots i_k}/\bar{H}_{i_1\dots i_k}$.

Thus we have proved:

THOREM. Let $\{i_1,\dots,i_k\}$ be a subset of the indices $\{1,2,\dots,\ell\}$ and let $S_{i_1,\dots,i_k}$ be the corresponding stable subvariety of $\bar{X}$. Let $P_{i_1\dots i_k}$ be the parabolic subgroup associated to the weight $\lambda_1 = \lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}$, then there is a G-equivariant fibration $\Pi_1 : S_{i_1,\dots,i_k} \to G/P_{i_1,\dots,i_k}$ with fibres isomorphic to the minimal compactification of $\bar{L}_{i_1\dots i_k}/\bar{H}_{i_1\dots i_k}$.

We should remark that in the ease of the "compactification of a group $\bar{G}$", the set $\{1,\dots,\ell\}$ can also be thought as the set of simple roots of G, for each subset the parabolic of $G \times G$ is $P \times P$ and the fiber of the $G \times G$ equivariant fibration is the "compactification of the adjoint group associated to the Levi factor of P".

5.3.

DEFINITION. $\bar{X}$ will be called simple if $\underline{g}$ = Lie G contains no proper σ-stable ideal.
It is clear that in this case either G is simple or we are in the case of a "compactification of a simple group".It also clear that in general $\bar{X}$ is the direct product of simple compactifications.

6. THE VARIETY OF LIE SUBALGEBRAS

6.1. We wish to compare our method with the one developed by Demazure in [5] and show that, in fact, his construction falls under our analysis.

The method is the following: consider the Lie algebras $\underline{g}$ and $\underline{h}$ of G, H respectively. Say dim $\underline{g} = n$, dim $\underline{h} = m$. Take for every $g \in G$ the subgroup gHg^{-1} and its Lie algebra ad$(g)\underline{h}$. The stabilizer in G of the subalgebra $\underline{h}$ under the adjoint action is exactly the subgroup $\tilde{H}$ considered in 2.1, so we can identify $G/\tilde{H}$ with the orbit of $\underline{h}$ in the Grassmann variety $G_{m,n}$ of m-dimensional subspaces in the n-dimensional space $\underline{g}$.

We define a compactification $\tilde{\underline{X}}$ of $G/\tilde{H}$ by putting $\tilde{\underline{X}} = \overline{G\underline{h}} \subseteq G_{n,m}$.

We want to show that $\tilde{\underline{X}}$ coincides with our $\bar{X}$. If we use the Plücker embedding, we see that we can identify $\tilde{\underline{X}}$ with the closure of the G-orbit of the point $\mathbb{P}(\overset{m}{\Lambda}\, \underline{h})$ in $\mathbb{P}(\overset{m}{\Lambda}\, \underline{g})$. If h is a vector spanning the line $\overset{m}{\Lambda}\, \underline{h}$, h is H invariant and we want to study its weight structure.

From Proposition 1.3 we know that

$$\underline{h} = \underline{t}_0 \oplus \sum_{\alpha\in\Phi_0} g_\alpha \oplus \sum_{\alpha\in\Phi_1^+} \mathbb{C}(x_\alpha + \sigma(x_\alpha))$$

so if

$$\{\beta_1,\dots,\beta_r\} = \Phi_0^+, \qquad \{\alpha_1,\dots,\alpha_t\} = \Phi_1^+$$

We have

$$\overset{m}{\Lambda}\, \underline{h} = \overset{k}{\Lambda}\, \underline{t}_0 \wedge x_{\beta_1} \wedge \dots \wedge x_{\beta_r} \wedge x_{-\beta_1} \wedge \dots \wedge x_{-\beta_r} \wedge (x_{\alpha_1} + \sigma(x_{\alpha_1}))$$

$$\wedge \dots \wedge (x_{\alpha_t} + \sigma(x_{\alpha_t})).$$

If we develop h and write it as a sum of weight vectors, we see that this sum contains a unique vector of weight $\mu = \alpha_1+\alpha_2+\dots+\alpha_t$ i.e. $\overset{k}{\Lambda} t_0 \wedge x_{\beta_1} \wedge \dots \wedge x_{\beta_r} \wedge x_{-\beta_1} \wedge \dots\dots \wedge x_{-\beta_r} \wedge x_{\alpha_1} \wedge \dots \wedge x_{\alpha_t}$ and the others have T_1 weitht of the form $\mu - 2\sum m_i\alpha_j$, $\alpha_j \in \Gamma_1$ and m_j non negative integers.

LEMMA. μ is a regular special weight.

PROOF. The fact that μ is special follows since $\mu^\sigma = -\mu$. To see that μ is regular recall that $2\rho = \beta_1 + \dots + \beta_r + \alpha_1 + \dots + \alpha_t$ and $(2\rho, \check{\alpha}_j) = 2$ while $(\beta_i, \check{\alpha}_j) \leq 0$ for each $\alpha_j \in \Gamma_1$ and $\beta_j \in \Phi_0^+$. Hence, clearly $(\mu, \check{\alpha}_j) \geq 2$ for each $\alpha_j \in \Gamma_1$.

We are now ready to deduce:

PROPOSITION. The compactification $\tilde{X} = \overline{G \cdot \underline{h}} \subseteq G_{m,n}$ is isomorphic to $\bar{X}$ of 2.1.

PROOF. Let $W \subset \overset{m}{\Lambda}\, \underline{g}$ be the minimum G-stable submodule containing $Ch = \overset{m}{\Lambda}\, \underline{h}$. Clearly for every irreducible component $V_i \subset W$ and G-equivariant projection $\Pi_i: W \to V_i$ we have $\Pi_i(h) \neq 0$.

In particular it follows from 1.5 that V_i has as its highest weight a special weight $\leq \mu$. Also, μ is a highest weight for W, we can now apply 4.1 and conclude the proof.

6.2. We can now easily see that the boundary points of $\tilde{X}$ are the Lie subalgebras (of groups related to the ones discussed in 6.2) as in Demazure's analysis.

In fact, to pass to the limit, up to conjugation, it is enough to do it under the action of T_1. If $t \in T_1$, we have:

$$t(\overset{m}{\Lambda}\, \underline{h}) = \overset{k}{\Lambda}\, \underline{t}_0 \wedge x_{\beta_1} \wedge \dots\dots \wedge x_{-\beta_r}$$

$$\wedge (x_{\alpha_1} + t^{-2\alpha_1}\sigma(x_{\alpha_1})) \wedge \dots \wedge (x_{\alpha_t} + t^{-2\alpha_t}\sigma(x_{\alpha_t}))$$

Going to the limit $t^{-2\alpha_i} \to 0$ if $i = i_1,\dots,i_k$ and $t^{-2\alpha_i} \to 1$ otherwise, we obtain the subalgebra spanned by

$$\underline{t}_0,\ x_{\beta_1},\dots,x_{\beta_r}, x_{-\beta_1},\dots,x_{-\beta_r}, x_{\alpha_k},\dots,x_{\alpha_j} + \sigma(x_{\alpha_j})$$

where k runs over all the indices for which α_k is a root of the unipotent radical $U_{i_1,\dots,i_k}$ of the parabolic $P_{i_1,\dots,i_k}$ and j runs over the remaining indeces.

This is the Lie algebra of the following subgroup. Consider the automorphism σ induced on $P_{i_1,\dots,i_k}/U_{i_1,\dots,i_k}$. Consider the fixed points of σ in $P_{i_1\dots i_k}/U_{i_1\dots i_k}$ and the subgroup of $P_{i_1,\dots,i_k}$ mapping onto this group of fixed points.

The Lie algebra is the one required by the previous analysis.

Remark that the projection from a G-orbit in $\bar{X}$ to the corresponding variety of parabolics is the one obtained by associating to a Lie algebra the normalizer of its unipotent radical.

7. COHOMOLOGY AND PICARD GROUP

7.1. We want now to describe a cellular decomposition of $\bar{X}$ which can be constructed, using the theory of Bialynicki-Birula [2], [26]. One of his main theorems is the following:

THEOREM. If $\bar{X}$ is a smooth projective variety with an action of a Torus T and if $\bar{X}$ has only a finite number of fixed points $\{x_1,\dots,x_n\}$ under T, one can construct a decomposition $\bar{X} = \cup C_{x_i}$ where each C_{x_i} is an affine cell (an affine space) centered in x_i.

The decomposition depends on certain choices. In particular, for a suitable choice of a one parameter group $\mu: G_m \to T$ such that $\bar{X}^{G_m} = \bar{X}^T$. Given such a choice, one decomposes the tangent space T_{x_i} of $\bar{X}$ at x_i as $T_{x_i} = T^+_{x_i} \oplus T^-_{x_i}$ (where T^+ and T^- are generated by vectors of positive respectively negative weight). Then C_{x_i} is an affine space of (complex) dimension dim $T^+_{x_i}$.

Furthermore, in [26], be shows that the variety $\bar{X}$ is obtained by a sequence of attachments of the C_{x_i}'s and so the integral homology has, as basis, the fundamental classes of the closures of the C_{x_i}'s (in particular it is concentrated in even dimensions and has no torsion).

7.2. In order to apply 7.1 we need the following proposition due to D. Luna.

PROPOSITION. Let G be a reductive algebraic group acting on a variety with finitely many orbits. If T is a maximal Torus of G, the set of fixed points X^T is finite.

PROOF. We can clearly reduce to the case in which X is itself an orbit. In this case it is enough to show that, if $x \in X^T$, x is an isolated fixed point. We have $X = Gx$ by assumption and $T \subseteq St_x$. The tangent space of X in x can be identified in a T equivariant way with Lie G/Lie St_x which is a quotient of Lie G/Lie T over which T acts without any invariant subspaces, proving the claim.

In particular we can apply this proposition to our variety $\bar{\underline{X}}$ in view of 3.1.

We should remark that in the case of a group G considered as $G \times G$ space, there are no fixed points on any non closed orbits. So the fixed points all lie in the closed orbit isomorphic to $G/B \times G/B$ and they are thus indexed by pairs of elements of the Weyl group.

7.3. Notice that, since $\bar{\underline{X}}$ has a paving by affine spaces, we have Pic $(\bar{\underline{X}}) \simeq H^2(\bar{\underline{X}})$. We want now to compute $H^2(\bar{\underline{X}})$ by computing the number of 2 dimensional cells given by 7.1.

For this we fix a Borel subgroup and the positive roots as in § 1. Since the center of G acts trivially on $\bar{\underline{X}}$, we can use the action of a maximal Torus T of the adjoint group. Hence, the simple roots are a basis of $\underline{t}^*$. We can construct a generic 1-parameter subgroup μ: $G_m \to T$ which has the same fixed points on $\bar{\underline{X}}$ as T and in the following way:

We order lexicographically the simple roots as

$$\beta_1 > \beta_2 > \dots \beta_k > \alpha_1 > \dots > \alpha_\ell > \alpha_{\ell+1} > \dots > \alpha_h$$

where $\bar{\alpha}_i = \frac{1}{2}(\alpha_i - \alpha_i^\sigma)$ $i = 1,\dots,\ell$ are the restricted simple roots.

We can, since in our computations there are only finitely many weights involved (the set Λ of weights appearing in the tangent spaces of the fixed points), select μ in such a way that $\langle\lambda,\mu\rangle > 0, \lambda \in \Lambda$ if and only if $\lambda > 0$ in the lexicographic ordering. If $x \in X$ is a fixed point of T, we analyze the tangent space τ_x as follows: x is in an orbit $\mathcal{O}$ which fibers Π: $\mathcal{O} \to G/P$ with fiber a symmetric variety $\bar{L}/\bar{L}^\sigma$, we can assume $x \in \bar{L}/\bar{L}^\sigma$ and decompose τ_x in T stable subspaces $\tau_1 \oplus \tau_2 \oplus \tau_3$ such that τ_1 is isomorphic to the tangent space of $\Pi(x)$ in $G/\underline{P}$, τ_2 is isomorphic to the tangent space of x in $\bar{L}/\bar{L}^\sigma$ and τ_3 is isomorphic to the normal space of $\mathcal{O}$ in $\bar{\underline{X}}$ at the point x. To compute dim τ^+ one needs

to compute dim τ_i^+ for each i. Now dim τ_1^+ is given by the theory of Bruhat cells, we claim:

LEMMA. 2 dim τ_2^+ = dim τ_2.

PROOF. The T-structure of τ_2 is isomorphic to the structure of the tangent space at the identity of $\bar{L}/\bar{L}^\sigma$ under the conjugate Torus $\bar{T} = x^{-1}Tx$. Such tangent space is isomorphic to $\underline{\ell}/\underline{\ell}^\sigma$ with ℓ = Lie $\bar{L}$, $\underline{\ell}^\sigma$ = Lie $\bar{L}^\sigma$. Since $\bar{T} \subset \bar{L}^\sigma$, we see that in the root space decomposition of $\underline{\ell}$ under $\bar{T}$ we have Lie $\bar{T} \subseteq \underline{\ell}^\sigma$. $\underline{\ell}^\sigma$ is a sum of root subspaces, and if $\underline{\ell}_\alpha \subset \underline{\ell}^\sigma$, also $\ell_{-\alpha} \subset \underline{\ell}^\sigma$. Thus, $\underline{\ell}/\underline{\ell}^\sigma$ is a sum of root spaces $\underline{\ell}_\beta \oplus \underline{\ell}_{-\beta}$. And then, if $\underline{\ell}_\beta \subset (\underline{\ell}/\underline{\ell}^\sigma)^+$, we have $\underline{\ell}_{-\beta} \subset (\ell/\underline{\ell}^\sigma)^-$ and the lemma is proved.

7.4. For the computation of the T weights in T_3 we have a simple analysis in the case in which the fixed point x lies in the closed orbit G/P.

In this case $x = w\,x_o$, w in the Weyl group and we have:

LEMMA. In $w\,x_o$ the dimension of τ_3^+ equals the number of restricted simple roots $\bar{\alpha}_i$ such that $w\bar{\alpha}_i > 0$.

PROOF. Using the notations of §.2, $x_o \in V \simeq U \times A^\ell$ and is identified with the point (1,0), $(1 \in U, 0 \in A^\ell)$. $G/P \cap V = U \times 0$, so the normal space at x_o is isomorphic to the space A^ℓ with the induced T-action.

Thus the normal space to a point $w\,x_o$ is isomorphic to $\mathbf{A}^\ell$ with the action twisted by w^{-1}. Since the T weights on A^ℓ are the $-2\bar{\alpha}_i$ we have that the T weights in the normal space at $w\,x_o$ are the elements $-2w\bar{\alpha}_i$, hence the claim.

7.5. In the computation of $H^2(X)$ we need to compute the points x such that dim $\tau_x^+ = 1$. Thus, we need in particular to analyze:

LEMMA. If G/H is a symmetric variety of dimension 2, with a fixed point under a Torus T', then Lie G = $s\ell(2)$, Lie H = $so(2)$ = Lie T', (up to normal factors on which the automorphism σ acts trivially).

PROOF: Let us recall the consequence of the Iwasawa decomposition 1.3.

$$\underline{g} = \underline{h} \oplus (\underline{t}_1 + \sum_{\alpha \in \Phi_1^+} \mathbb{C}\, x_\alpha); \quad \text{Thus, } 2 = \dim \underline{t}_1 + |\Phi_1^+|.$$

Since we generally have $\underline{t}_1 \neq 0$ if $G/H \neq 1$ and also $|\Phi_1^+| \neq 0$ since G is semisimple, we must have $1 = \dim \underline{t}_1 = |\Phi_1^+|$. Moreover, since we want to factor out all normal subgroups of G on which σ acts trivially, we have G simple. We wish to show that Φ_0 is empty. In fact, if there is a simple root $\beta \in \Phi_0$, since G is simple we may assume that $\beta + \alpha$ is also a root. But then either β or $\beta+\alpha \in \Phi_1^+$ and we have a contradiction. Then we see that G is of rank 1 and the remaining statements easily follow.

7.6. We are now ready for the computation of Pic (X).

THEOREM. Pic (X) $\cong Z^{\ell+r}$ where r is the number of simple roots α_i, $i = 1,\dots,\ell$ such that: there exist two distinct simple roots α, β with $\bar{\alpha}_i = \frac{1}{2}(\alpha - \alpha^\sigma) = \frac{1}{2}(\beta - \beta^\sigma)$ and either $-\alpha^\sigma \neq \beta$ or, if $-\alpha^\sigma = \beta$, $(\alpha,\beta) \neq 0$.

PROOF. Let $\bar{\alpha}_i = \frac{1}{2}(\alpha_i - \alpha_i^\sigma)$, $i = 1,\dots,\ell$ be the simple restricted roots (cf. 2.2). Suppose $x \in \bar{X}$ is a fixed point with $\dim \tau_x^+ = 1$, first of all we analyze the case in which $x \in G/P$, the unique closed orbit. In this case $\tau_x = \tau_1 + \tau_3$ and we must have either $\dim \tau_1^+ = 0$, $\dim \tau_3^+ = 1$ or $\dim \tau_1^+ = 1$, $\dim \tau_3^+ = 0$. Now x is a center of a Bruhat cell in G/P of dimension equal to $\dim \tau_1^+$ so it is either the point x_o corresponding to the 0 cell or a point $s_\alpha x_o$ with α a simple root in Φ_1^+. Thus, by Lemma 7.4 $\dim \tau_3^+$ at wx_o is the number of i such that $w\bar{\alpha}_i$ is negative. In particular we see that we can get 2 dimensional cells only centered at the points $s_\alpha x_o$ and we need to count how many $\alpha \in \Phi_1^+$ are such that $s_\alpha \bar{\alpha}_i > 0$ for all i's. Now if $\alpha \neq \alpha_i, -\alpha_i^\sigma$, we have $s_\alpha(\bar{\alpha}_i) > 0$ (since $s_\alpha(\beta) > 0$ if β is positive $\alpha \neq \beta$). Now given $\alpha \in \Phi_1^+$ if $\alpha = \alpha_i$, we have $s_\alpha(\bar{\alpha}_j) > 0$ if $j \neq i$. As for $s_\alpha(\bar{\alpha}_i)$ it depends on $-\alpha_i^\sigma$.

We have various cases:

i.) $-\alpha_i^\sigma = \alpha_i$,

ii.) $-\alpha_i^\sigma = \alpha_i + \beta$, $\beta \neq 0$ a positive combination of roots in Φ_0.

iii.) $-\alpha_i^\sigma = \alpha_j + \beta$, $j \neq i$.

In case i.) $s_\alpha(\bar{\alpha}) = -\bar{\alpha} < 0$,

In case ii.) $s_\alpha(\bar{\alpha}) = -\alpha + \frac{1}{2}(\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha) > 0$,

In case iii.) the same reasoning as in ii.) holds if $\beta \neq 0$,

$s_\alpha(\alpha + \alpha_j + \beta) = \beta + \alpha_j + m\alpha > 0$ (some m).

If $\beta = 0$, we have

$$s_\alpha(\alpha + \alpha_j) = -\alpha + \alpha_j - \frac{2(\alpha,\alpha_j)}{(\alpha,\alpha)}\alpha$$

Now since $\alpha_j = -\alpha^\sigma$, we must have $(\alpha,\alpha) = (\alpha_j,\alpha_j)$. Hence, the Dynkin diagram formed by α, α_j is either disconnected and $(\alpha,\alpha_j) = 0$ or is A_2 and then $\frac{2(\alpha,\alpha_j)}{(\alpha,\alpha)} = -1$ so $s_\alpha(\alpha + \alpha_j) = \alpha_j > 0$. If $(\alpha,\alpha_j) = 0$, we have $s_\alpha(\alpha + \alpha_j) = -\alpha + \alpha_j < 0$ since $\alpha = \alpha_i$ $i \leq \ell$ and $j > \ell$.

Now we have to consider the case $\alpha = -\alpha_i^\sigma \neq \alpha_i$, since α is a simple root this occurs only in the case $-\alpha_i^\sigma = \alpha_j$, $j > \ell$. The same analysis as before shows that

if $(\alpha,\alpha_i) = -\frac{1}{2}$ we have $s_\alpha(\alpha + \alpha_i) = \alpha > 0$

if $(\alpha,\alpha_i) = 0$ $\quad s_\alpha(\alpha + \alpha_i) = \alpha_i - \alpha > 0$.

It remains to analyze the case of x lying in a non closed orbit $\mathcal{O}$. By Lemmas 7.3 and 7.5 this can occur only when $\mathcal{O}$ fibers on a variety G/P' with fiber the minimal compactification of a symmetric variety isomorphic to $SL(2)/S\mathcal{O}(2)$. This is the variety of distinct unordered pairs of points in $\mathbb{P}^1$ and its minimal compactification is the space $\mathbb{P}^2$ considered as the symmetric square of $\mathbb{P}^1$. In this case we only have 2 SL(2) orbits in $\mathbb{P}^2$ and so only 2 G orbits in $\bar{\mathcal{O}}$.

Thus by 3.1 we have $\dim \bar{\mathcal{O}} = \dim G/P + 1$ and a $\mathbb{P}^1$-fibration $G/P \to G/P'$. Thus, we can identify P' with the parabolic group generated by P and the subgroup $X_{-\alpha}$ relative to a simple root $\alpha \in \Phi_1^+$ and we have $\alpha^\sigma = -\alpha$, and $\alpha = \alpha_i$ for some $1 \leq i \leq \ell$. As in Lemma 7.3 write $\tau_x = \tau_1 \oplus \tau_2 \oplus \tau_3$. Since T acts on τ_2 by a negative and a positive weight as we have noted above in order to have that the set of T weights appearing on τ_x contains only a positive weight, we must have that the T weights in τ_1 and τ_3 consist of negative weights. This implies that $p(x) \in G/P'$ is the unique B fix point in G/P', otherwise at least one of the weights appearing in τ_1 would be positive. Furthermore, notice that the fact that p(x) is the unique B fix point in G/P' determines x uniquely since in $p^{-1}(p(x)) = \mathbb{P}^2$ there are exactly three T fix points of which two are x_o and $s_\alpha(x_o)$ both belonging to the closed orbit. But for such x we have that the set of weights appearing on τ_3 is $\{\frac{(\alpha_j - \alpha_j^\sigma) + s_\alpha(\alpha_j - \alpha_j^\sigma)}{2}\}$ for $1 \leq j \leq \ell$, $j \neq i$ which are all negative.

This is easily seen as follows: first of all the normal bundle to $\bar{\mathcal{O}}$ in X is just the sum of the restrictions of the normal line bundles to the closures of the codimension one orbits S_j, $1 \leq j \leq \ell$, $j \neq i$, containing $\bar{\mathcal{O}}$. Thus we have to compute the weight of T for each such line bundle N_j. Let us fix $1 \leq j \leq \ell$, $j \neq i$, then the T weight of N_j in x_o is just $-(\alpha_j, -\alpha_j^\sigma)$. Now if we let $T_\alpha \subset T$ denote ker α, we have that T_α acts trivially on $\mathbb{P}^2$ hence the T_α weight in x and x_o are the same. Thus the given formula is correct for T_α. It remains to verify the formula on a "complement of T_α in T. This amounts to perform the computation in the maximal torus of PSL(2) which can be carried out directly.

So it follows that the action of T on τ_x has exactly one negative weight and the cell associated to x has dimension 2. Summarizing our result we have

1) If $\bar{\alpha}_i$ is such that there exists only one simple root α with $\frac{1}{2}(\alpha - \alpha^\sigma) = \bar{\alpha}_i$ and $\alpha^\sigma \neq -\alpha$ then we get one 2 cell whose center lies in the unique closed orbit G/P.
2) If α_i is as in one but $\alpha^\sigma = -\alpha$ then again we get one 2 cell but its center lies in the orbit $\mathcal{O}$ whose closure $\bar{\mathcal{O}}$ fibers with $\mathbb{P}^2$ fibers onto G/P', P' being the parabolic generated by P and $X_{-\alpha}$.
3) If $\bar{\alpha}_i$ is such that there exists two distinct simple roots α,β such that $\bar{\alpha}_i = \frac{1}{2}(\alpha - \alpha^\sigma) = \frac{1}{2}(\beta - \beta^\sigma)$, $-\alpha^\sigma = \beta$ and $(\alpha,\beta) = 0$ then we get exactly one 2 cell whose center lies in G/P.
4) If $\bar{\alpha}_i$ is such that $\bar{\alpha}_i = \frac{1}{2}(\alpha - \alpha^\sigma) = \frac{1}{2}(\beta - \beta^\sigma)$ and either $-\alpha^\sigma \neq \beta$ or $-\alpha^\sigma = \beta$ but $(\alpha,\beta) \neq 0$, then we get two 2 cells, whose both centers lie in G/P.

This is our theorem.

DEFINITION. $\underline{\bar{X}}$ will be called exceptional when rk Pic$(\underline{\bar{X}}) > \ell$.

7.7. REMARK. It is clear from the previous analysis that the main difficulty in computing explicitely the dimensions of the cells lies in the computation of τ_3^+. In the special case in which all fixed points lie in the closed orbit this is accomplished by Lemma 7.4.

In particular for the case of a group $\bar{G}$ considered as a symmetric variety over $\bar{G} \times \bar{G}$ we have the following computation for the Poincarè polynomial: $\Sigma b_i q^j$, $b_i = \dim H_i(\underline{\bar{X}},\mathbb{Z})$:

$$\Big(\sum_{w\in W} q^{2\ell(w)}\Big)\Big(\sum_{w\in W} q^{2(\ell(w)+L(w))}\Big) \; (*)$$

($\ell(w)$ the length of w, L(w) the number of simple reflections s_α with $\ell(s_\alpha w) < \ell(w)$).

8. LINE BUNDLES ON $\underline{\bar{X}}$

8.1. Let $\underline{\bar{X}}$ be as usual and let $Y = G/P \subset \underline{\bar{X}}$ be the unique closed orbit in $\underline{\bar{X}}$.

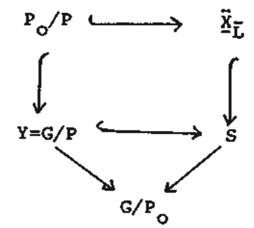
PROPOSITION. Let i^*: Pic $(\underline{\bar{X}}) \to$ Pic (Y) be the homomorphism induced by the inclusion. Then i^* is injective.

PROOF. First assume that for any simple root $\alpha \in \Phi_1^+$ we have $\alpha^\sigma = -\alpha$. Then we know that Pic $(\underline{\bar{X}}) \cong \mathbb{Z}^\ell$, where ℓ is the number of simple roots in Φ_1^+. Furthermore, let $\omega_1,\ldots,\omega_\ell$ be the fundamental weights corresponding to such α's. Then we have shown how to imbed $\underline{\bar{X}} \subset \prod_{i=1}^{s} \mathbb{P}(V_{2\omega_i})$. So we get a map h^*: Pic$(\prod_{i=1}^{s} \mathbb{P}(V_{2\omega_i})) \to$ Pic $(\underline{\bar{X}})$. But it is clear that i^*h^* is injective since the restriction of the tautological bundle L_i on $\mathbb{P}(V_{2\omega_i})$ to G/P gives the line bundle associated to $2\omega_i$. Since rk(Pic$(\prod_{i=1}^{s} \mathbb{P}(V_{2\omega_i}))$) = rk(Pic (X)) our assertion follows.

(*) We wish to thank G. Lusztig for suggesting this formula.

Let us now suppose that there exists a simple root α such that $\alpha^\sigma \neq -\alpha$. Let S be the unique orbit closure associated to $\alpha - \alpha^\sigma$. Then it follows from the description of the dimension two cells given in 7, that each dimension 2 cell in $\underline{\bar{X}}$ is already contained in S, so we prove that the map Pic $(\underline{\bar{X}}) \to$ Pic (S) induced by inclusion is injective.

Let us now consider the map Pic (S) $\to$ Pic (Y) and recall that for a suitable parabolic P_o we get a fibration $S \to G/P_o$ whose fiber is the variety $\underline{\bar{X}}_{\bar{L}}$ which is the minimal compactification of $\bar{L}/\bar{L}^\sigma$ where $\bar{L}$ is the adjoint group of the semisimple Levi factor of P_o and $\bar{L}^\sigma$ the fix points group of the involution induced by σ on $\bar{L}$. We thus get the diagram



and we can identify P/P_o with the unique closed orbit in $\underline{\bar{X}}_{\bar{L}}$. But notice that Pic $(G/P) \cong$ Pic $(G/P_o) \oplus$ Pic (P/P_o) and Pic $(S) \cong$ Pic $(G/P_o) \oplus$ Pic $(\underline{\bar{X}}_{\bar{L}})$. Also, by induction on the rank we can assume that the map Pic $(\underline{\bar{X}}_L) \to$ Pic (P/P_o) induced by inclusion is injective. This clearly implies that the map Pic (S) $\to$ Pic (Y) is also injective.

REMARK. Notice that since we can identify Pic (Y) with the lattice spanned by the fundamental weights relative to the simple roots in Φ_1^+, our proposition implies that we can also identify Pic (X) with a sublattice of such a lattice, call it Γ. Notice also that since for each dominant special weight λ with the property that $\frac{2(\lambda,\alpha-\alpha^\sigma)}{(\alpha-\alpha^\sigma,\alpha-\alpha^\sigma)} \in \mathbb{Z}^+$ for every simple root $\alpha \in \Phi_1^+$ we have constructed a map Π: $X \to \mathbb{P}(V_\lambda)$ we clearly have that Γ contains the lattice spanned by such weights. In particular, this lattice contains the double of the lattice of special weights $\alpha - \alpha^\sigma \in \Gamma$ for each simple root $\alpha \in \Phi_1^+$.

We wish to collect some of the information gotten up to now for future use.

We have the weights μ_i introduced in 1.7 and a natural embedding

$$\bar{X} \to \Pi\mathbb{P}(V_{\mu_i})$$

The mapping of the closed orbit $Y \to \Pi\mathbb{P}(V_{\mu_i})$ so induced is the canonical one obtained by the diagonal morphism. We compose this with the natural projection $G/B \to Y$.

The ample generator of Pic $(\mathbb{P}(V_{\mu_i}))$ is mapped by the composed homomorphism to the element L_{μ_i} of Pic(G/B) corresponding to the weight μ_i (notice that under this convention $H^o(G/B,L_{\mu_i}) \cong V^*_{\mu_i}$ as a G-module).

If J is a subset of $\{1,\dots,\ell\}$ and S_J denotes the corresponding orbit closure, the composition $S_J \to \bar{X} \to \Pi\mathbb{P}(V_{\mu_i}) \xrightarrow{P_J} \prod_{i\in J}\mathbb{P}(V_{\mu_i})$ factors through the canonical fibration $S_J \to G/P_J$ and the canonical inclusion $G/P_J \hookrightarrow \prod_{i\in J}\mathbb{P}(V_{\mu_i})$. Therefore in particular the line bundle corresponding to μ_i restricted to S_J comes from the corresponding line bundle in G/P_J.

Finally since Pic$(\bar{X})$ is discrete and G is simply connected any $L \in$ Pic $(\bar{X})$ has a G linearization ([27]). Suppose now $L \in$ Pic $(\bar{X})$ is a G linearized line bundle. If we restrict this to the closed orbit Y we have the induced bundle already linearized. Now for a linearized line bundle L_λ on Y the corresponding weight λ is the character by which the maximal torus acts on the fiber over the unique B fix point, x_o, in Y.

Recall that the cell $U \times \mathbb{A}^\ell$ in $\bar{X}$ is a B^- stable affine subspace and $(1,0)$ is the fixed point x_o in Y previously introduced. If δ is a section trivializing L_λ on $U \times \mathbb{A}^\ell$ so is $b*\delta$ for any $b \in B^-$. Since the only invertible functions on $U \times \mathbb{A}^\ell$ are the constants we have $b*\delta = \alpha\delta$, α a scalar. Restricting to the point x_o we have $\alpha = b^{-\lambda}$.

8.2. Notice that since any $L \in$ Pic (X) can be G linearized we have that G acts linearly on each $H^i(\bar{X},L)$.

LEMMA. Let $L \in$ Pic $(\bar{X})$ and consider $H^o(\bar{X},L)$ as a G module. Then $\dim \mathrm{Hom}_G(V,H^o(\bar{X},L)) \leq 1$ for each irreducible G-module V.

PROOF. Suppose $\mathrm{Hom}_G(V,H^o(\bar{X},L)) \neq 0$. Let μ be the highest weight of V. Let $s_1,s_2 \in H^o(\bar{X},L)$ be two non zero U invariant sections whose weight is μ. Then $\frac{s_1}{s_2}$ is a B invariant rational function on $\bar{X}$. Since B has a dense orbit in $\bar{X}$, it follows the $\frac{s_1}{s_2}$ is constant. Hence, s_1 is a multiple of s_2 and our claim follows.

Now let $V \subset \bar{X}$ be the open set described in 2 and identify V with

$U \times \mathbb{A}^\ell$. Let $\{x_i\}$ be the coordinate functions on $\mathbb{A}^\ell$. For any $t \in T$, $tx_i = t^{-(\alpha_i-\alpha_i^\sigma)}x_i$ for the corresponding simple root $\alpha_i \in \Phi_1^+$, $1 \leq i \leq \ell$.

PROPOSITION. Let V_μ be the irreducible G-module whose highest weight is μ. Let $\lambda \in \Gamma$ and $L_\lambda \in$ Pic $(\bar{X})$ be the corresponding line bundle, then if

$$\mathrm{Hom}(V^*_\mu,H^o(\bar{X},L_\lambda)) \neq 0 \qquad \mu = \lambda - \sum t_i(\alpha_i - \alpha_i^\sigma), \quad t_i \in \mathbb{Z}^+$$

PROOF. Let $s \in H^o(\bar{X},L_\lambda)$ be a section generating a B^- stable line. Then if we restrict s to V and we let s_o be a section trivializing $L_\lambda|V$ we can write $s = s_o f$ where f is a regular function on $V \cong U \times \mathbb{A}^\ell$. Since s is U stable f is also U stable and $f = x_1^{t_1},\dots,x_\ell^{t_\ell}$ so our proposition follows.

COROLLARY. There exists a unique up to a scalar G-invariant section $r_i \in H^o(\bar{X},L_{\alpha_i-\alpha_i^\sigma})$ whose divisor is S_i.

PROOF. Let $r_i \in H^o(\bar{X},\mathcal{O}(S_i))$ be the unique, up to constant, section whose divisor is S_i. Since S_i is G-stable and G is semisimple, r_i is a G-invariant section. Also since $x_i = 0$ is a local equation of S_i on V we have $r_i|V = s_o x_i$ where s_o is a section trivializing $\mathcal{O}(S_i)|V$. The weight of x_i is $\alpha_i - \alpha_i^\sigma$ so the G-invariance of r_i implies that s_o has weight $-(\alpha_i - \alpha_i^\sigma)$. Hence $\mathcal{O}(S_i) \cong L_{\alpha_i - \alpha_i^\sigma}$.

8.3. Now let $S_{\{i_1,\dots,i_t\}} = S_{i_1} \cap \dots \cap S_{i_t}$ for any subset $\{i_1,\dots,i_t\} \subset \{1,\dots,\ell\}$ be the corresponding G-stable subvariety. Let $\gamma \in \Gamma$ put $L_\gamma(i_1,\dots,i_t) = L_\gamma|S_{i_1,\dots,i_t}$. Let $\{j_1,\dots,j_{\ell-t}\}$ denote the complement in $\{1,\dots,\ell\}$ of $\{i_1,\dots,i_t\}$.

PROPOSITION. Let $\gamma \in \Gamma$ be a dominant weight. Let $\{h_1,\dots,h_s\} \subset \{j_1,\dots,j_{\ell-t}\}$. Then

$$H^i(S_{\{i_1,\dots,i_t\}}, L_{\gamma-\Sigma(\alpha_{h_i}-\alpha_{h_i}^\sigma)}(i_1,\dots,i_t)) = 0 \quad \text{for} \quad i > 0.$$

PROOF. We perform a double decreasing induction on $\{i_1,\dots,i_t\}$ and on $\{h_1,\dots,h_s\}$.

If $\{i_1,\dots,i_t\} = \{1,\dots,\ell\}$ then $(1,\dots,\ell) = G/P$ is the unique closed orbit and our proposition is part of Bott's theorem [4].

Now let $\{i_1,\dots,i_t\}$ be arbitrary and $\{j_1,\dots,j_{\ell-t}\} = \{h_1,\dots,h_s\}$. Then notice that by our local description of $\bar{X}$ it follows easily that if $K(i_1,\dots,i_t)$ denotes the canonical bundle on $S_{\{i_1,\dots,i_t\}}$, $K(i_1,\dots,i_t) = L_{-\mu-\sum_{m=1}^{\ell-t}(\alpha_{j_m} - \alpha_{i_m}^\sigma)}(i_1,\dots,i_t)$ where $\mu = \sum_{\alpha\in\Phi_1^+}\alpha$.

(Notice that $\mu \in \Gamma$ (cf. 6.1)).

Thus if we put $L = L_{\gamma-\Sigma(\alpha_{j_m}-\alpha^{\sigma}_{j_m})}(i_1,\dots,i_t)$ and $K = K(i_1,\dots,i_t)$ we have that $(K \otimes L^{-1})^{-1} = L_{\gamma+\mu}(i_1,\dots,i_t)$ can be verified to be very ample. We postpone the proof of this assertion to the end of this section. It follows from Kodaira vanishing theorem that

$$H^i(S_{\{i_1,\dots,i_t\}}, K \otimes L^{-1}) = 0 \quad \text{for} \quad i < \dim S_{i_1,\dots,i_t}.$$

This implies by Serre's duality

$$H^i(S_{\{i_1,\dots,i_t\}}, L) = 0 \quad \text{for } i > 0.$$

Now by induction we have the result proved for any $S_{\{i_1,\dots,i_{t+1}\}}$ and for any $\{h_1,\dots,h_{s+1}\} \subset \{j_1,\dots,j_{\ell-t}\}$.

Corollary 8.2 implies that we have a non zero section

$$r_{i_{t+1}} \in H^0(S_{\{i_1,\dots,i_t\}}, L_{\alpha_{i_{t+1}}-\alpha^{\sigma}_{i_{t+1}}}(i_1,\dots,i_t))$$

and multiplication by $r_{i_{t+1}}$ yields an exact sequence.

$$0 \to L_{\gamma - \sum_{i=1}^{s}(\alpha_{h_i} - \alpha^{\sigma}_{h_i} - (\alpha_{i_{t+1}} - \alpha^{\sigma}_{i_{t+1}}))}(i_1,\dots,i_t) \to$$

$$L_{\gamma - \sum_{i=1}^{s}(\alpha_{h_i} - \alpha^{\sigma}_{h_i})}(i_1,\dots,i_t) \to L_{\gamma - \sum_{i=1}^{s}(\alpha_{h_i} - \alpha^{\sigma}_{h_i})}(i_1,\dots,i_{t+1}) \to 0$$

Then we get a long exact sequence that together with an inductive hypothesis immediately proves the proposition.

THEOREM. Let $\lambda \in \Gamma$ then:

1) $H^0(\bar{X}, L_\lambda) \neq 0$ if and only if $\lambda = \gamma + \Sigma t_i(\alpha_i - \alpha^{\sigma}_i)$ for some dominant γ, $t_i \in \mathbb{Z}^+$. Assuming $H^0(X, L_\lambda) \neq 0$, if V_γ is the irreducible G-module of highest weight γ, $H^0(X, L_\lambda) = \oplus V^*_\gamma$ for all dominant γ of the form $\gamma = \lambda - \Sigma t_i(\alpha_i - \alpha^{\sigma}_i)$, $t_i \in \mathbb{Z}^+$.

2) For λ dominant $H^i(\bar{X}, L_\lambda) = 0$, $i > 0$.

PROOF.

1) The only if part is just Proposition 8.2.

To prove the if part assume λ is dominant. Then we know that $H^0(G/P, L_\lambda|_{G/P})$ is the irreducible G-module V_λ whose highest weight is λ. Now consider the varieties

$$\bar{X} = S_\phi \supset S_{\{1\}} \supset S_{\{1,2\}} \supset S_{\{1,2,3\}} \supset \dots\dots S_{\{1,2,\dots,\ell\}} = G/P$$

We claim that for each $\ell \geq i \geq 1$ the restriction map

$$H^0(S_{\{1,2,\dots,i-1\}}, L_\lambda|S_{\{1,2,\dots,i-1\}}) \to H^0(S_{\{1,2,\dots,i\}}, L_\lambda|S_{\{1,2,\dots,i\}})$$

is onto.

This follows at once from the cohomology exact sequence associated to the sequence

$$0 \to L_{\lambda-(\alpha_i-\alpha^{\sigma}_i)}(1,2,\dots,i-1) \to L_\lambda(1,2,\dots,i-1) \to L_\lambda(1,2,\dots,i) \to 0$$

considered above and the vanishing of

$$H^1(S_{\{1,\dots,i-1\}}, L_{\lambda-(\alpha_i-\alpha^{\sigma}_i)}(1,2,\dots,i-1))$$

proved in Proposition 8.3.

In particular, the restriction map

$$H^0(\bar{X}, L_\lambda) \to H^0(G/P, L_\lambda|_{G/P}) \quad \text{is onto.}$$

Hence, $\mathrm{Hom}_G(V^*_\lambda, H^0(\bar{X}, L_\lambda)) \neq 0$ and we can find a non zero lowest weight vector $v_\lambda \in H^0(\bar{X}, L_\lambda)$ whose weight is $-\lambda$.

Now let $\lambda = \gamma + \sum_{i=1}^{\ell} t_i(\alpha_i - \alpha^{\sigma}_i)$, $t_i \in \mathbb{Z}^+$, γ dominant in Γ.

Consider the section $r_1^{t_1} \dots\dots r_\ell^{t_\ell} \in H^0(\bar{X}, L_{\sum_{i=1}^{\ell} t_i(\alpha_i - \alpha^{\sigma}_i)})$ and the section $v_{-\gamma} \in H^0(\bar{X}, L_\gamma)$.

Then the section $v_{-\gamma} r_1^{t_1} \dots r_\ell^{t_\ell}$ is clearly non zero U-invariant and its weight is $-\gamma$. So $\mathrm{Hom}_G(V_\gamma, H^0(\bar{X}, L_\lambda)) \neq 0$. This proves 1); 2) is contained in Proposition 8.3.

REMARK.

1) By a completely analogous argument we can prove that if $\lambda \in \Gamma$ then

$$\mathrm{Hom}\,(V^*_\gamma, H^0(S_{\{i_1,\dots,i_t\}}, L_\lambda|S_{\{i_1,\dots,i_t\}})) \neq 0$$

if and only if

$$\lambda = \gamma + \sum_{m=1}^{\ell-t} t_m(\alpha_{j_m} - \alpha_{j_m})$$

2) Clearly we can define a filtration of $H^0(\bar{X}, L_\lambda)$ by putting for each ℓ-tuple of non negative integers $(t_1,\dots,t_\ell)$, $W(t_1,\dots,t_\ell)$ to be the subspace of sections $s \in H^0(\bar{X}, L_\lambda)$ vanishing on S_1 of order $\geq t_1,\dots,$ on S_ℓ of order $\geq t_\ell$. Then we can restate our theorem as follows:

$$W_\lambda(t_1,\dots,t_\ell) \Big/ \sum_{(\bar t_1,\dots,\bar t_\ell) > (t_1,\dots,t_\ell)} W_\lambda(\bar t_1,\dots,\bar t_\ell) \cong \begin{cases} 0 & \text{if } \lambda-\sum t_i(\alpha_i-\sigma(\alpha_i)) \text{ is not dominant} \\ V^*_{\lambda-\sum t_i(\alpha_i-\sigma(\alpha_i))} & \text{otherwise} \end{cases}$$

(Here $(\bar t_1,\dots,\bar t_\ell) \geq (t_1,\dots,t_\ell)$ means $\bar t_1 \geq t_1,\dots,\bar t_\ell \geq t_\ell$).

8.4. In order to complete the proof of 8.3 we have to discuss the ampleness of $L_{\gamma+\mu}$ $(i_1 \dots i_t)$ which has been used there.

We start with a general easy fact. Let ω,ω' denote two distinct fundamental weigths V_ω, $V_{\omega'}$, $V_{\omega+\omega'}$ the irreducible representations of highest weight $\omega,\omega',\omega+\omega'$.

We have a canonical G equivariant projection $p: V_\omega \otimes V_{\omega'} \to V_{\omega+\omega'}$ and we denote by $\bar p$ the induced projection $\mathbb{P}(V_\omega \otimes V_{\omega'}) \to \mathbb{P}(V_{\omega+\omega'})$ of projective spaces: Remark that $\mathbb{P}(V_\omega) \times \mathbb{P}(V_{\omega'})$ is embedded in $\mathbb{P}(V_\omega \otimes V_{\omega'})$ via the Segre map.

LEMMA. The map $\bar p$ restricted to $P(V_\omega) \times P(V_{\omega'})$ is a regular embedding.

PROOF. We consider the irreducible representations of G as sections of line bundles an G/B so that the map p corresponds to the usual multiplication. Since G/B is irreducible the product of 2 non zero sections is always non zero. Now if $s,s' \in V_\omega$, $t,t' \in V_{\omega'}$ and $st = s't'$ we claim that $s' = cs$, $t' = c^{-1}t$, c a scalar. In fact since ω,ω' are fundamental the divisors of s,s',t,t' are all irreducible since ω,ω' are independent in Pic (G/B) the divisor of s cannot equal the divisor of t' and so we have divs = divs' and the claim.

This proves that $\bar p$ is injective when restricted to $P(V_\omega) \times P(V_{\omega'})$. To see that the map is also smooth one can use the same fact in local affine coordinates.

We are now ready to prove:

PROPOSITION. For any $\gamma \in \Gamma$ dominant the line bundle $L_{\gamma+\mu}$ is ample on $\underline{X}$ hence also on $S_{(i_1,\dots,i_t)}$ for any choice of $i_1,\dots,i_t$.

PROOF. We distinguish 2 cases. If γ is special, since μ is a regular special weight so is $\mu+\gamma$ hence by 3.1 and 4.1 we have that $L_{2(\mu+\gamma)}$ is very ample on X.

Assume γ not special. This can happen only if we are in the exceptional case i.e. if the rkPic$(\underline{X}) > \ell$ since if a multiple of a weight γ is special so is γ and Pic$(\underline{X})$ contains the double of the lattice of special weights.

First of all we can clearly reduce to the case is which X is simple (cf. 5.3).

In the group case $\underline{X} = \overline{G \times G/G}$ we have rk Pic$(\underline{X})$= rkG = ℓ by remark 7.7 otherwise $\underline{X} = G/\hat{H}$ with G simple.
We know by 7.6 that rk Pic$(\underline{X}) > \ell$ if and only if there exists a simple root α such that:

$$\alpha^\sigma = -\alpha' - \beta \quad \text{with} \quad \alpha' = \alpha \quad \text{and either } \beta \neq 0 \text{ or } (\alpha^\sigma,\alpha') \neq 0.$$

Now we can inspect the tables of Satake diagrams in the classification of symmetric spaces (cf. [10], p. 532-534) and we see using the notations of such tables that the only cases to be considered are the ones denoted by AIII (first diagram) A IV, D III (second diagram), EIII. One remarks by inspecting the table V (p. 518) that these cases belong to table III (p. 515).

In all cases one can verify that there is a unique pair of simple roots α,α' with the above properties and hence rk Pic$(\underline{X})$= $\ell+1$.

Case AIII and AIV can be explicitely described as follows.

We consider in $\underline{s\ell}_n$ the automorphism σ defined as conjugation by the block matrix

$$\begin{pmatrix} I_k & 0 \\ e & -I_{n-k} \end{pmatrix} \quad \text{with } k \neq n-k.$$

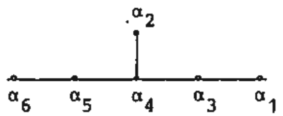
Case DIII can be described as
so(4n+2) relative to the symmetric form

$$\begin{pmatrix} 0 & I_{2n+1} \\ I_{2n+1} & 0 \end{pmatrix}$$

and conjugation relative to

$$\begin{pmatrix} I_{2n+1} & 0 \\ 0 & -I_{2n+1} \end{pmatrix}$$

For E III consider the Dynkin diagram of E_6 indexed as



relative to a Cartan subalgebra $\underline{t}$.

Denote by x_α the generator of the corresponding root subspace and define σ as the identity on $\underline{t}$,

$$\sigma(x_{\alpha_i}) = x_{\alpha_i}, \quad i \neq 1, \quad \sigma(x_{\alpha_1}) = -x_{\alpha_1}.$$

One can now verify in each case that the fixed group H is the intersection of a suitable maximal parabolic subgroup Q of type α with its opposite Q' which in all cases is of type α'.

Let us denote by ω and ω' the dual fundamental weights to α, α' and V_ω, V^*_ω, the corresponding irreducible representations. We remark that $V_\omega \simeq V_{\omega'}$ and by 1.3 that $\omega^\sigma = -\omega'$, so that $\omega + \omega'$ is a special weight. If $v \in V_\omega$ (resp. $v' \in V_{\omega'}$) generate the line fixed by Q (resp. by Q') we have that v, v' are seminvariants under H and $v \otimes v'$ is an H invariant, thus if we project $v \otimes v'$ on $V_{\omega+\omega'}$ we obtain a non zero H invariant. By the analysis of section 4 we have a regular morphism π of $\bar{X}$ onto the orbit closure Y of the class of $v \otimes v'$ in $\mathbb{P}(V_{\omega+\omega'})$.

We show now that Y is isomorphic to $G/Q \times G/Q'$. This follows from Lemma 8.4 in the following way. In $\mathbb{P}(V_\omega \otimes V_{\omega'})$ the $G \times G$ orbit of $v \otimes v'$ is clearly $G/Q \times G/Q'$ and this orbit projects isomorphically to its image in $\mathbb{P}(V_{\omega+\omega'})$ under $\bar{p}$. On the other hand an easy computation of dimensions shows that the G orbit of $v \otimes v'$ is open in $G/Q \times G/Q'$ hence its closure is $G/Q \times G/Q'$. Since $\bar{p}$ is G-equivariant everything is proved. Comparing the map $\bar{X} \to Y \simeq G/Q \times G/Q'$ with the two projections and the respective Plücker embeddings we have two regular projective morphisms associated to the non special weights ω, ω'. We go back now to γ and claim that a suitable positive multiple of γ is of the form $\zeta + a\omega$ or $\zeta + a\omega'$ with $a > 0$ and ζ a dominant special weight.

This can be shown remarking that the subgroup Γ' of Γ generated by the special weights and ω has the same rank as Γ thus a positive multiple of γ lie in Γ'. Now if a dominant weight is in Γ', using the notations of 1.3 it is of the form

$$m\gamma = \sum n_i \omega_i + a\omega \quad \text{with} \quad n_i = n_{\tilde{\sigma}(i)},$$

and ω (resp. ω') is one of the ω_i's, for istance $\omega = \omega_1$ (resp. $\omega' = \omega_2$).

Also $m\gamma$ being dominant $n_1 + a \geq 0$ and $n_i \geq 0$ for $i \neq 1$. If $a \geq 0$ we are done otherwise

$$m\gamma = (n_1 + a)(\omega + \omega') + \sum_{i>2} n_i \omega_i - a\omega'.$$

From this it is clear that for any dominant $\gamma \in \Gamma$ the complete linear system associated to a suitable positive multiple of γ is without base points, since μ is very ample this implies that $\mu + \gamma$ is ample.

9. COMPUTATION OF THE CHARACTERISTIC NUMBERS

9.1. In section 7 we have computed Pic $(\bar{X}) \simeq H^2(\bar{X},\mathbb{Z})$. We want now to give an explicit algorithm to compute the characteristic numbers. This means that, given n elements $x_1,\ldots,x_n \in H^2(\bar{X},\mathbb{Z})$, $n = \dim \bar{X}$, we wish to evaluate the product $x_1 \ldots x_n \in H^{2n}(\bar{X},\mathbb{Z})$ against the class of a point.

Given n reduced hypersurfaces $D_1,\ldots,D_n$ in $\bar{G}/\bar{H}$ such that their closures in $\bar{X}$, $\bar{D}_i$ do not contain the unique closed orbit, if $x_i = \mathcal{O}(\bar{D}_i) \in \text{Pic}(\bar{X}) \simeq H^2(\bar{X},\mathbb{Z})$ the corresponding characteristic number counts exactly the number of points common to generic translates $g_i D_i$, $g_i \in G$, of the D_i's (this is an easy consequence of [12] since $\bar{X}$ has a finite number of orbits).

We may work in $H^2(\bar{X},\mathbb{Q})$ and use suitable bases for this space. We may also assume that $\bar{X}$ is simple (cf. 5.3).

It follows from the analysis performed in section 8 that Pic $(\bar{X}) \otimes \mathbb{Q}$ can be identified with the vector space generated by the special weights if $\bar{X}$ is not exceptional, otherwise one has to add to the special weights a fundamental weight ω.

Let us denote with Σ the vector space spanned by the special weights and, in the exceptional case $\Gamma_{\mathbb{Q}} = \Sigma + \mathbb{Q}\omega$.

We also know that the divisors S_i correspond to twice the restricted simple roots and form a basis of Σ. Denote by $[S_i]$ these elements in Σ. We have another basis of Σ given by the elements λ_j (cf. 4.1). We notice that $(\lambda_j,[S_i]) = 0$ if $i \neq j$ (for the Killing form).

LEMMA. If $i_1,\ldots,i_k$, $j_1,\ldots,j_{\ell-k}$ is a shuffle of the indices $1,2,\ldots,\ell$, the elements $\lambda_{i_1},\ldots,\lambda_{i_k},[S_{j_1}],\ldots,[S_{j_{\ell-k}}]$ form a basis of Σ.

PROOF. Clear by the orthogonality relations.

9.2. Given an oriented compact manifold X and an oriented submanifold Y denote by [Y] the Poincaré dual of the fundamental class of Y. We shall use the following basic facts:

1) If Y_1, Y_2 are oriented submanifolds of X with transversal intersection we have:

$$[Y_1 \cap Y_2] = [Y_1] \cup [Y_2]$$

2) If $Y \subset X$ is a d-dimensional oriented submanifold and $c \in H^d(X)$ we have that the evaluation of $c \cup [Y]$ on the class of a point in X equals the evaluation of $c|_Y$ on the class of a point in Y.

The main proposition is the next one.

PROPOSITION. Let $S_{\{i_1 \dots i_k\}} = S_{i_1} \cap \dots \cap S_{i_k}$. If $S_{\{i_1 \dots i_k\}}$ is not the closed orbit in $\bar{X}$ then:

1) Every monomial $\lambda_{i_1}^{h_1} \lambda_{i_2}^{h_2} \dots\dots \lambda_{i_k}^{h_k}$ with $\Sigma h_i = \dim S_{\{i_1 \dots i_k\}}$ vanishes on $S_{\{i_1 \dots i_k\}}$.

2) In the exceptional case every monomial $\omega^{h_o} \lambda_{i_1}^{h_1}, \lambda_{i_2}^{h_2} \dots\dots \lambda_{i_k}^{h_k}$ with $\Sigma h_i = \dim S_{\{i_1 \dots i_k\}}$ vanishes on $\tilde{S}_{\{i_1 \dots i_k\}}$.

PROOF. 1) Recall that we have a projection $\pi: S_{\{i_1 \dots i_k\}} \to G/P_{\{i_1 \dots i_k\}}$ and the classes $\lambda_{i_1}, \dots, \lambda_{i_k}$ come via π^* from the cohomology of $G/P_{\{i_1 \dots i_k\}}$. Since $S_{\{i_1 \dots i_k\}}$ is not the closed orbit we have $\dim S_{\{i_1 \dots i_k\}} > \dim G/P_{\{i_1 \dots i_k\}}$ and everything follows.

2) We have seen in 8.4 that L_ω induces a morphism $p: \bar{X} \to G/Q$ for a suitable maximal parabolic Q and ω is the pullback of the ample generator of Pic (G/Q) by p^*. We wish to consider the induced map $\pi \times p: S_{\{i_1 \dots i_k\}} \to G/P_{\{i_1 \dots i_k\}} \times G/Q$ and denote by $\tilde{S}_{\{i_1 \dots i_k\}}$ its image. We know that $\omega + \omega^\sigma$ is one of the fundamental special weights λ_i. If the index i is one of the indeces of the set $\{i_1, \dots, i_k\}$ then the parabolic Q contains $P_{\{i_1 \dots i_k\}}$ and the projection $p: S_{\{i_1 \dots i_k\}} \to G/Q$ factors through $G/P_{\{i_1 \dots i_k\}}$. This case therefore follows as in 1). Otherwise $G/P_{\{i_1 \dots i_k\}} \times G/Q$ contains a unique closed orbit under G isomorphic to $G/P_{\{i_1 \dots i_k\}} \cap Q$. We claim that $\tilde{S}_{\{i_1 \dots i_k\}}$ equals this orbit. In fact first of all the fiber of the projection $G/P_{\{i_1 \dots i_k\}} \cap Q \to G/P_{i_1 \dots i_k}$ equals the variety $L_{\{i_1 \dots i_k\}}/L_{\{i_1 \dots i_k\}} \cap Q$ which is a complete homogeneous space over the semisimple part of $L_{\{i_1 \dots i_k\}}$.

If we restrict to a fiber $X_{\{i_1 \dots i_k\}}$ of π the line bundle L_ω we obtain a line bundle of the same type (relative to the minimal compactification $X_{\{i_1 \dots i_k\}}$ of $\bar{L}_{\{i_1 \dots i_k\}}/\bar{H}_{\{i_1 \dots i_k\}}$ (cf. 5.2)).
Since we know that $H^o(X_{\{i_1 \dots i_k\}}, L_\omega | X_{\{i_1 \dots i_k\}})$ is an irreducible $L_{\{i_1 \dots i_k\}}$ module we get that the restriction homomorphism

$$H^o(X, L_\omega) \to H^o(X_{\{i_1 \dots i_k\}}, L_\omega | X_{\{i_1 \dots i_k\}})$$

is onto. Hence the induced morphism on $X_{\{i_1 \dots i_k\}}$ coincides with the restriction to $X_{\{i_1 \dots i_k\}}$ of p and maps it onto $L_{\{i_1 \dots i_k\}}/L_{\{i_1 \dots i_k\}} \cap Q$. This proves the claim. Since $S_{i_1 \dots i_k}$ is not the closed orbit $\dim \tilde{S}_{\{i_1 \dots i_k\}} < \dim S_{\{i_1 \dots i_k\}}$ and everything follows as in 1.

9.3. We are now ready to illustrate the algorithm. We treat the exceptional case, the non exceptional is the same without the appearence of ω.

Consider monomials of degree n of type $M = [S_{i_1}] \dots [S_{i_k}] \omega^{h_o} \lambda_{j_1} \dots \lambda_{j_s}$ with $i_1, \dots, i_k$ distinct (in particular the ones with $k = 0$ are the monomials we wish to evaluate). We call k the index of M. We count the number of indices j_h appearing in M and different from $i_1, i_2, \dots, i_k$ and call this the content of M.

If $j_1 \neq i_1, i_2, \dots, i_k$ we have an explicit formula expressing λ_{j_1} in terms of $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k}$ and the $[S_j]$'s relative to the remaining indeces (Lemma 9.1).

Substituting we obtain M expressed as a linear combination of monomials of higher index and of lower content.

Iterating we obtain M as a combination of monomials of index ℓ or of content 0.

By Proposition 9.2 all monomials of contenent 0 vanish, the computation of the remaining ones can be performed:

LEMMA. The evaluation of $[S_1][S_2] \dots [S_\ell] \omega^{h_o} \lambda_{i_1} \dots \lambda_{i_k}$ on the class of a point in X equals the evaluation of $\omega^{h_o} \lambda_{i_1} \dots \lambda_{i_k}$ restricted to the closed orbit on the class of a point in it.

PROOF. Clear since the closed orbit is the transversal intersection of the hypersurfaces S_i.

We summarize

THEOREM. By an explicit algorithm the computation of the characteristic numbers is reduced to the one relative to the closed orbit (for which it is known since the cohomology ring of a complete homogeneous space is known [3]).

10. AN EXAMPLE

10.1. In his fundamental work [14] H. Schubert has computed the number of space quadrics tangent to 9 quadrics in general position to be 666.841.088. We want here to perform again this computation.

The variety of non degenerate quadrics in $\mathbb{P}^n$ is symmetric, it is $X_o = SL(n+1)/\hat{SO}(n+1)$ (the involution being $\sigma(A) = {}^tA^{-1}$).

The variety X is classically called the variety of complete quadrics ([1],[15],[17],[19],[21],[22]).

One can easily verify (by the invariant theory of the orthogonal group) that the irreducible representations of SL(n+1) containing an invariant for SO(n+1) are exactly the ones of highest weight $\sum_{i=1}^{n} n_i 2\omega_i$ (ω_i the fundamental weights). From this it follows that we can identify Pic $(\bar{X})$ with 2Λ where Λ is the lattice of weights for SL(n+1) and that the closed orbit in $\bar{X}$ is the full flag variety F. The usual maximal Torus of diagonal matrices is anisotropic and so the restricted simple roots coincide with the usual simple roots. Hence:

$$[S_1] = 2(2\omega_1) - 2\omega_2$$
$$[S_i] = 2(2\omega_i) - 2\omega_{i-1} - 2\omega_{i+1} \qquad 1 < i < n$$
$$[S_n] = 2(2\omega_n) - 2\omega_{n-1}.$$

Let us fix for each $i = 0,\dots,n-1$ a linear subspace π_i of dimension i in $\mathbb{P}^n$. Denote by D_i the hypersurfaces in X_o of quadric tangent to π_i. We also fix a non degenerate quadric Q and denote by D the hypersurface in X_o of quadrics tangent to Q. We denote as usual by $\bar{D}_i$, $\bar{D}$ their closures in $\bar{X}$.

PROPOSITION.
1) $[\bar{D}_i] = \mathcal{O}(\bar{D}_i) = L_{2\omega_i}$.
2) $[\bar{D}] = 2\sum_{i=0}^{n-1}[\bar{D}_i]$
3) $\bar{D}_i$ and $\bar{D}$ do not contain the closed orbit.

PROOF. 1) X_o is the affine variety of symmetric $(n+1)\times(n+1)$ matrices of determinant 1. The map from X_o to $\mathbb{P}(V^*_{2\omega_i})$ is easily seen to be induced by the map associating to each matrix the matrix of determinants of $i\times i$ minors, which gives a quadric in $\mathbb{P}(V_{\omega_i})$ whose intersection with the Grassmann variety $G_{i-1,n}$ of i-1 dimensional subspaces is exactly the set of tangent subspaces to the original quadric.

Given an i-1 dimensional subspace π_{i-1} in $\mathbb{P}^n$ we consider it as a point in $G_{i-1,n}$, hence, by taking the embedding of $G_{i-1,n}$ in $\mathbb{P}(V_{2\omega_i})$ as a point in $\mathbb{P}(V_{2\omega_i})$. Then it is clear that the intersection of X_o with the hyperplane in $\mathbb{P}(V^*_{2\omega_i})$ associated to this point is at least set theoretically D_{i-1}. So we have found an $s \in H^0(\bar{X}, L_{2\omega_i})$ whose divisor has support equal to D_{i-1}. But it is clear from our computation of Pic (X) that the divisor of s is reduced so it equals $\bar{D}_{i-1}$ proving 1).

2) Consider the variety $F_{0,n-1}$ of flags $p \in \pi \subset \mathbb{P}^n$ where p is a point and π is an hyperplane. Define a flag (p,π) to be tangent to a quadric $Q \in \bar{X}_o$ if $p \in Q$ and π is the hyperplane tangent to Q in p. Let $Y \subset \bar{X}\times F_{0,n-1}$ be the closure of the correspondence $\tilde{Y} = \{(Q,(p,\pi)) \mid (p,\pi)$ is tangent to Q, $Q \in \bar{X}_o\}$. Clearly $\dim Y = \dim \bar{X} + n - 1 = \frac{(n+1)(n+2)}{2} + n - 2$ and we get two projections

$$\bar{X} \xleftarrow{\;g\;} Y \xrightarrow{\;f\;} F_{0,n-1}$$

A simple dimension count shows that we have an homomorphism

$$g_*f^*: H^n(F_{0,n-1},\mathbb{Z}) \to H^2(\bar{X},\mathbb{Z})$$

Consider our complete flag $\pi_o \subset \pi_1 \subset \quad \subset \pi_{n-1} \subset \mathbb{P}^n$.
It is well known that a basis of $H^n(F_{0,n-1},\mathbb{Z})$ is given by the classes dual to the following Schubert subvarieties:

$$Y_i = \{(p,\pi) \mid p \subset \pi_i \subset \pi\}.$$

On the other hand it follows easily from our definition of Y that $g_*f^*([Y_i]) = [\bar{D}_i]$ so that g_*f^* is an isomorphism.

Furthermore if we fix a quadric $Q \in \bar{X}_o$ and we embed it in $F_{0,n-1}$ by associating to each point in Q its tangent flag we get that $g_*f^*([Q]) = [\bar{D}]$ so that in order to prove our claim it is sufficient to show that

$$[Q] = \sum_{i=0}^{n-1} 2[Y_i] \quad \text{in} \quad H^2(F_{0,n-1},\mathbb{Z})$$

Denote by $Y'_o,\dots,Y'_{n-1}$ the Schubert cycles dual to $Y_o,\dots,Y_{n-1}$; i.e. $Y'_i = \{(p,\pi) \mid p \subset \pi_{n-i},\ \pi \supset \pi_{n-i-1}\}$. We are reduced to show that the evaluation on the class of a point in $F_{0,n}$ of $[Q]\cdot[Y_i]$ is 2 for each $0 \le i \le n-1$. This is clear by elementary considerations on the geometry of quadrics.

3) We first show that $\bar{D}_i \not\supset F$ for each $0 \leq i \leq n-1$. Assume the contrary and let $s \in H^0(\bar{X}, L_{2\omega_i})$ be a section whose divisor is $\bar{D}_i$. The restriction of s to F is zero. On the other hand it follows from our results of section 8 that the restriction homomorphism

$$j^*: H^0(\bar{X}, L_{2\omega_i}) \to H^0(F, L_{2\omega_i}|F)$$

is an isomorphism.

We now show our result for $\bar{D}$. For this, given a non singular quadric $Q \in X_o$, define a flag $f \in F$ to be tangent to Q if the point of f lies in Q and the hyperplane of f is the hyperplane tangent to Q in this point. Consider the variety $Z \subset \bar{X} \times F$ which is the closure of the correspondence $\tilde{Z} = \{(Q,f) | Q \in X_o,\ f$ is tangent to $Q\}$. Consider the fibration $p: \bar{X} \times F \to \bar{X} \times F_{0,n-i}$ induced by the natural fibration $q: F \to F_{0,n-i}$. Then we claim $Z = p^{-1}(Y)$. This is clear since $\tilde{Z} = p^{-1}(\tilde{Y})$. This allows us to determine the fiber of the projection $g: Z \to \bar{X}$ over a point f_o in the closed orbit.

In fact think of f_o as a flag $f_o = \{\pi_o \subset \pi_1 \subset \ldots \subset \pi_{n-1} \subset \mathbb{P}^n\}$ and for each $f \in g^{-1}(f_o)$ put $q(f) = (p, \pi)$. We claim that $g^{-1}(f_o) = \cup Z_i$, where $Z_i = \{f | p \subset \pi_i \subset \pi\}$.

To see this notice that the image of f_o in $\mathbb{P}(V^*_{2\omega_i})$ under the morphism $\bar{X} \to \mathbb{P}(V^*_{2\omega_i})$ represents a degenerate quadric in $\mathbb{P}(V_{\omega_i})$ whose intersection with the Grassmannian of $i-1$ dimensional subspaces is just the set of such subspaces intersecting π_{n-1}.

Thus if $f \in g^{-1}(f_o)$ its $(i-1)$ dimensional subspace has to meet π_{n-i}. In particular $p \in \pi_{n-1}$.

Assume $p \in \pi_i - \pi_{i-1}$. We claim $\pi \supset \pi_i$. In fact if $i \geq 1$ each $(n-i)$ dimensional subspace τ with $p \in \tau \subset \pi$ has to meet π_{i-1} by the above remarks, and if $i = 0$ there is nothing to prove. So $f \in Z_i$. Having shown this it is easily seen that given f_o in the closed orbit of $\bar{X}$ such that π_i is not tangent to Q for all $0 \leq i \leq n-1$, $f_o \notin \bar{D}$ proving 3).

COROLLARY. The evaluation at the class of a point of any monomial of the form

$$(2\omega_1)^{h_1} \cdots\cdots (2\omega_n)^{h_n} \Big(2 \sum_{i=1}^{n} 2\omega_i\Big)^{h_{n+1}}$$

with $\sum_{i=1}^{n+1} h_i = \frac{(n+1)(n+2)}{2} - 1 = \dim \bar{X}$ gives the number of quadrics which are simultaneously tangent to h_1 points, h_2 lines,......, h_n hyperplanes, h_{n+1} quadrics lying in general position.

REMARK. Our proof of the fact that $\bar{D} \not\supset F$ works also in the case in

which $\bar{D}$ is the closure in $\bar{X}$ of the hypersurface of X_o of quadrics tangent to any fixed subvariety in $\mathbb{P}^n$. Thus since $[\bar{D}]$ can be written as a linear combination of the $[\bar{D}_i]$'s the problem of enumerating the number of quadrics simultaneously tangent to $\frac{(n+1)(n+2)}{2} - 1$ subvarieties in general position is reduced to the same problem for linear spaces. This fact has been recently shown in a much greater generality by Fulton, Kleiman, Mac Pherson.

In the case of $\mathbb{P}^3$ working out the computations with the algorithm given in 9.2 one finds the following table which can also be found in Schubert's book (p. 105):

$x_1^9 = x_3^9 = 1$	$x_1^6x_2^2x_3 = x_3^6x_2^2x_1 = 12$
$x_1^8x_2 = x_3^8x_2 = 2$	$x_1^5x_2^3x_3 = x_3^5x_2^3x_1 = 24$
$x_1^7x_2^2 = x_3^7x_2^2 = 4$	$x_1^4x_2^4x_3 = x_3^4x_2^4x_1 = 48$
$x_1^6x_2^3 = x_3^6x_2^3 = 8$	$x_1^6x_2x_3^2 = x_3^6x_2x_1^2 = 18$
$x_1^5x_2^4 = x_3^5x_2^4 = 16$	$x_1^5x_2^2x_3^2 = x_3^5x_2^2x_1^2 = 36$
$x_1^4x_2^5 = x_3^4x_2^5 = 32$	$x_1^4x_2^3x_3^2 = x_3^4x_2^3x_1^2 = 72$
$x_1^3x_2^6 = x_3^3x_2^6 = 56$	$x_1^5x_2x_3^3 = x_3^5x_2x_1^3 = 34$
$x_1^2x_2^7 = x_3^2x_2^7 = 80$	$x_1^4x_2^2x_3^3 = x_3^4x_2^2x_1^3 = 68$
$x_1x_2^8 = x_3x_2^8 = 92$	$x_1^4x_2x_3^4 = 42$
$x_2^9 = 92$	$x_1^2x_2^6x_3 = x_3^2x_2^6x_1 = 104$
$x_1^8x_3 = x_3x_1^8 = 3$	$x_1^3x_2^5x_3 = x_3^3x_2^5x_1 = 80$
$x_1^7x_3^2 = x_3^2x_1^7 = 9$	$x_1^3x_2^4x_3^2 = x_3^3x_2^4x_1^2 = 112$
$x_1^6x_3^3 = x_3^3x_1^6 = 17$	$x_1x_2^7x_3 = 104$
$x_1^4x_3^5 = x_3^5x_1^4 = 21$	$x_1^2x_2^5x_3^2 = 128$
$x_1^7x_2x_3 = x_3^7x_2x_1 = 6$	$x_1^3x_2^3x_3^3 = 104$

and so

$$(2(x_1 + x_2 + x_3))^9 = 666.841.088$$

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GEOMETRIC INVARIANT THEORY AND APPLICATIONS TO MODULI PROBLEMS

D. Gieseker
University of California
Los Angeles, California 90024 USA

These notes are a brief introduction to geometric invariant theory (GIT) a contain two applications of that theory to the construction of moduli spaces in gebraic geometry. The first two sections sketch the basics of GIT over the com numbers. In §3 we connect GIT and the theory of stable bundles of rank two on non-singular curve. We then consider in §4 the relation between smooth curves GIT. Our main result here is that there are enough projective invariants of sp curves to separate any two projectively distinct smooth curves of genus g and degree d provided $d \geq 2g$ and that the curves are non-degenerate. This resul can be used to construct a moduli space $\mathfrak{M}_g$ for smooth curves of genus g. In sections §5 and §6, we look at the connection between stable curves in the sense of Mumford and Deligne and stable curves in the senses of GIT. The main resul essentially that the compactification $\overline{\mathfrak{M}}_g$ of $\mathfrak{M}_g$ considered by Mumford and De is a projective variety. (This result was originally obtained by F. Knutsen in characteristic zero using other methods.) Finally in §7 we indicate how GIT can used to construct compactified generalized Jacobians of stable curves. Here we c sider the example of an irreducible curve with one node. The nature of the compactification of the generalized Jacobian of a general stable curve obtained by GIT has yet to be worked out. One can also extend the results of §5,6,7 to vect bundles of rank two [G-M,G_4]. Roughly, one gets a construction of a projectiv moduli space of stable bundles on an irreducible curve which has one node. This can then be used to study the topology of the moduli space of stable bundles on smooth curve by degeneration methods.

The original source for the first two sections is [M_1], but [N] also pro vides a more leisurely treatment. A connection between GIT and the theory of stable bundles on a smooth curve was worked out by Mumford and Seshadri. [N] conta an account of this work. In these notes, we make a slightly different connection which is more suitable for higher dimensional varieties [G_1,Ma]. Mumford gave a proof of the existence of $\mathfrak{M}_g$ using GIT in [M_1] using the Chow variety of a space curves. Here we use Grothendieck's Hilbert scheme which is arguably easi [G_2] contains an extension of these ideas to the n canonical images of surface of general type. The connection between GIT and stable curves was worked out jointly by Mumford and myself using the Chow variety and Hilbert scheme [M_2,G_3]. Finally an exhaustive discussion of the developments in GIT since the first edition of Mumford's book and the present can be found in the second edition of Mumford's book.